SAT and Hybrid models of the Car-Sequencing problem

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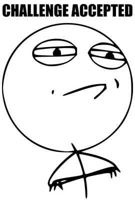
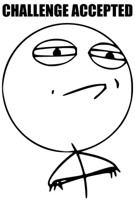
Outline

How did it start?

Mohamed	Valentin
CP lover.. We have Global Constraints!	The SAT-revolution.. Encode Finite domain CSP into SAT !
Hybrid SAT/CP techniques	'advanced' SAT encoding

The Car-sequencing problem





CP/SAT Solving

SAT & CP :

- to encode into SAT or to use global constraints?
- Can we get the best from both approaches?
  →A key concept in hybrid solvers : explaining constraints

An explanation is a set of assignments/prunings triggering a failure/filtering.

example

Cardinality Constraint : $\sum_{i=1}^{n} x_i \leq k$; $D(x_i) = \{0, 1\}$.
$x_i \leftarrow 1$ is pruned if we already have k appearances of the value 1.

$$\{x_j \leftarrow 1 | D(x_j) = \{1\}\} \rightarrow x_i \not\leftarrow 1 .$$

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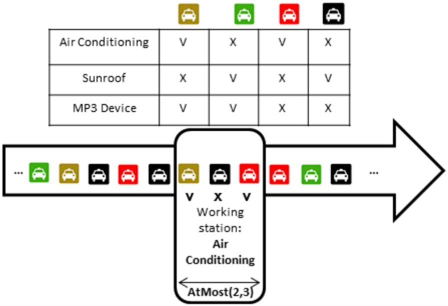

Constraints

- Each class c is associated with a demand D_c.
- For each option j, each sub-sequence of size q_j must contain at most u_j cars requiring the option j.

Modelling in CP

Variables :

- n integer variables $\{x_1, \dots, x_n\}$ taking values in $\{1, \dots, k\}$
- nm Boolean variables $\{y_1^1, \dots, y_n^m\}$

Constraints:

1. *Demand constraints* : for each class $c \in \{1..k\}$

$$|\{i \mid x_i = c\}| = D_c^{class}.$$

$$\rightarrow \textsc{Gcc}$$

2. *Capacity constraints* : for each option $j \in \{1..m\}$, for each slot $i \in \{1, \dots, n - q_j + 1\}$.

$$\sum_{l=i}^{i+q_j-1} y_l^j \leq u_j.$$

$\rightarrow$ Gsc, AtMostSeqCard or AtMostSeqCard$\oplus$ Gsc

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Definition

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$\textsc{AtMostSeqCard}(u, q, d, [x_1, \ldots, x_n]) \Leftrightarrow$

$$\bigwedge_{i=0}^{n-q} \left(\sum_{l=1}^{q} x_{i+l} \leq u\right) \wedge \left(\sum_{i=1}^{n} x_i = d\right)$$

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Example $\textsc{AtMostSeqCard}(2, 4, 4, [x_1, \ldots, x_7])$

0 1 1 0 1 1 0

1 1 0 0 1 0 1

The propagator

- leftmost: computes an assignment w maximizing the cardinality of the sequence with respect to the AtMost constraints.
- Let $max(i)$ be the maximum cardinality of the q subsequences involving x_i when computing leftmost[i].

0 1 0 . 0 1 0

$cardinality = 1$

$cardinality = 0$

$cardinality = 1$

$max(4) = 1$

- $Left[i] = \sum_{j=1}^{j=i} \texttt{leftmost}[j]$.
- $Right[i]$: same as *Left* but in the reverse sense, i.e. $[x_n, .., x_1]$.
- Example : with AtMost(2,5):

$\mathcal{D}(x_i)$	0	.	.	.	.	1	.	.	
$max(i)$	0	1	2	2	2	2	1	2	2
$\texttt{leftmost}[i]$	0	1	0	0	0	1	1	0	0
$Left[i]$	0	1	1	1	1	1	2	2	2

Domain consistency

- DC on each AtMost: $(\sum_{l=1}^{q} x_{i+l} \leq u)$
- DC on $\sum_{i=1}^{n} x_i = d$
- If $Left[n] < d$ Then *fail*
- If $Left[n] = d$ and $Left[i] + Right[n-i+1] \leq d$ Then $\mathcal{D}(x_i) \leftarrow \{0\}$
- If $Left[n] = d$ and $Left[i-1] + Right[n-i] < d$ Then $\mathcal{D}(x_i) \leftarrow \{1\}$

Explaining AtMostSeqCard: the key idea

Explaining Failure

1. If a failure is triggered by a cardinality constraint (i.e. $(\sum_{l=1}^{q} x_{i+l} \leq u)$ or $\sum_{i=1}^{n} x_i = d$), then it is easy to generate an explanation.
2. If a failure triggered by $Left[n] < d$, a naive explanation would be the set of all assignments in the sequence.

Some observations

Let S : 1 1 0 0 . subject to ATMOST(2/5).
→`leftmost` on S gives 1 1 0 0 0

Consider the sequence S_0 : 1 1 . 0 .
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Consider the sequence S_2 : . 1 0 0 .
→`leftmost` on S_2 gives 1 1 0 0 0

$$\{x_i \leftarrow 1 \mid max(i) \neq u\}$$

Theorem

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Let S be the set of all assignments,
$S^{\cdot *} = S \setminus (\{x_i \leftarrow 0 \mid max(i) = u\} \cup \{x_i \leftarrow 1 \mid max(i) \neq u\})$*, then*
$S^{\cdot *}$ *is a valid explanation.*

→runs in $O(n)$ since we call `leftmost` once.

Example : AtMostSeqCard$(2, 5, 8, [x_1, ..x_{22}])$

S	1	0	1	0	0	.	.	0	0	0	1	1	0	0	0	0	1	0	0	0	0	1
$\texttt{leftmost}(S(x_i))$	1	0	1	0	0	1	0	0	0	0	1	1	0	0	0	0	1	0	0	0	0	1
$Left[i]$	1	1	2	2	2	3	3	3	3	3	4	5	5	5	5	5	6	6	6	6	6	7

$Left[22] = 7 < 8$: FAILURE

$max(i)$	2	2	2	2	2	1	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1
$S^{\cdot *}$	1	.	1	.	.	.	.	.	.	.	1	1	.	.	.	0	.	0	0	0	0	.

The final explanation size $|S^{\cdot *}|$ is 9 while the naive one ($|S|$) is 20.

Explaining pruning

explanation for $x \leftarrow k$?

1. Add $x \not\leftarrow k$ to the instantiation where the pruning was performed.
2. Use the previous procedure to explain the failure on the new instantiation.

PB & SAT Modelling

Variables:

- c_i^j : c_i^j is *true* iff the class of the ith slot is j.
- y_i^j : y_i^j is *true* iff the ith vehicle requires option j.

Constraints:

- Demand constraints : $\forall j \in [1..k], \sum_i c_i^j = D_j$
- Capacity constraints : $\sum_{l=i}^{i+q_j-1} y_l^j \leq u_j$
- Channelling:
  - $\forall i \in [1..n], \forall l \in [1..k]$, we have :
    - $\forall j \in \mathcal{O}_l, \overline{c_i^l} \vee y_i^j$
    - $\forall j \notin \mathcal{O}_l, \overline{c_i^l} \vee \overline{y_i^j}$
  - a redundant clause : $\forall i \in [1..n], j \in [1..m], \overline{y_i^j} \vee \bigvee_{l \in \mathcal{C}_j} c_i^l$
- $\forall i \in [1..n], \sum_j c_i^j = 1$

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SAT model? encode Cardinality constraints: Sequential counter, Cardinality Networks, Sorting network, etc.

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- $\forall i \in [1..n], \sum_j c_i^j = 1$

SAT model? encode Cardinality constraints: Sequential counter, Cardinality Networks, Sorting network, etc.

Sequential Counter (SC) [Sin05]

Encoding $\sum_{i \in [1..n]} x_i = d$ to a CNF ?

- Variables:
  - $s_{i,j}$: $\forall i \in [0..n], \forall j \in [0..d+1]$, $s_{i,j}$ is *true* iff $\sum_{k \in [1..i]} x_k \geq j$
- Encoding: $\forall i \in [1..n]$
  - Clauses for restrictions on the same level: $\forall j \in [0..d+1]$
    1. $\neg s_{i-1,j} \vee s_{i,j}$
    2. $x_i \vee \neg s_{i,j} \vee s_{i-1,j}$
  - Clauses for increasing the counter, $\forall j \in [1..d+1]$
    3. $\neg s_{i,j} \vee s_{i-1,j-1}$
    4. $\neg x_i \vee \neg s_{i-1,j-1} \vee s_{i,j}$
  - Initial values for the bounds of the counter:
    5. $s_{0,0} \wedge \neg s_{0,1} \wedge s_{n,d} \wedge \neg s_{n,d+1}$

Example $\sum_{i\in[1..8]} x_i = 2$

3	0	0	0	0	0	0	0	0	0
2	0	0	.	.	.	.	.	.	1
1	0	.	.	.	.	.	.	1	1
0	1	1	1	1	1	1	1	1	1
$s_{i,j}$	0	1	2	3	4	5	6	7	8
x_i		.	.	.	.	.	.	.	.

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x_i		.	.	.	.	.	.	.	.

3	0	0	0	0	0	0	0	0	0
2	0	0	.	.	.	.	.	.	1
1	0	.	1	1	1	1	1	1	1
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1	0	.	1	1	1	1	1	1	1
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1	0	0	1	1	1	1	1	1	1
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1	0	.	.	.	.	.	.	1	1
0	1	1	1	1	1	1	1	1	1
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x_i		.	.	.	.	.	.	.	.

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2	0	0	.	.	.	.	.	.	1
1	0	.	1	1	1	1	1	1	1
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x_i		.	1	.	.	.	.	.	.

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x_i		0	1	0	0	0	0	1	0

$\rightarrow$Unit Propagation on the SC encoding enforces AC on the cardinality constraint $\sum_{i\in[1...n]} x_i = d$.

Extension to AtMostSeqCard

AtMostSeqCard: AtMost⊕ Cardinality→[SC] !

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Using a similar encoding of the Gen-Sequence constraint [Bac07, BNQ+07] [SCS].

7. $\neg s_{i,j} \vee s_{i-q,j-u}$

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8 $\neg s_{i,j} \vee s_{i-q,j-u}$

Proposition

The level of pruning using (SCS) is incomparable with SC on each AtMost(SCA).

Configuration

- SAT :
  1. *SAT (1)* SC
  2. *SAT (2)* SCS
  3. *SAT (3)* SC $\oplus$ SCS.
- Mistral as a hybrid CP/SAT solver
  1. *Hybrid (VSIDS)*
  2. *Hybrid (Slot)*
  3. *Hybrid (Slot $\rightarrow$ VSIDS)*
  4. *Hybrid (VSIDS $\rightarrow$ Slot)*
- *pseudo Boolean*: MiniSat+
- *CP*: [Mistral]

Table: Evaluation of the models

Method	sat[easy] (74 × 5)			sat[hard] (7 × 5)			unsat/unknown (28 × 5)		
	#suc	avg fails	time	#suc	avg fails	time	#suc	avg fails	time
SAT (1)	370	2073	1.71	28	337194	282.35	**85**	**249301**	**105.07**
SAT (2)	370	1077	1.18	30	42790	33.02	67	217103	182.23
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Hybrid (VSIDS)	370	903	0.23	16	207211	286.32	35	177806	224.78
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CP	**370**	**43.06**	**0.03**	35	57966	16.25	0	-	-
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- Finding solutions quickly :
  - CP-based models are difficult to outperform!
  - Overall, the best method on satisfiable instances is the hybrid solver using a pure CP heuristic.
  - with *VSIDS*, MiniSat on the strongest encodings has good results!
  - Propagation is very important to find solutions quickly when they exist, by keeping the search “on track” and avoiding exploring large unsatisfiable subtrees.

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- For proving unsatisfiability
  - Clause learning is by far the most critical factor.
  - Surprisingly, the “lightest” encoding gave best results!

Conclusion & Future research

Contributions

- First non-trivial SAT encoding for the car-sequencing problem.
- A linear time explanation for the AtMostSeqCard constraint
- Closing 13 out of the 23 large open instances.

Future research

- Can we generate optimal explanations for AtMostSeqCard?
- Other SAT-encoding for the Cardinality constraint?
- Optimisation problems?

Thank you!

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