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Rotation-Based Formulation for Stable Matching

Mohamed Siala, Barry O'Sullivan

November 10, 2017

Matching Under Preferences

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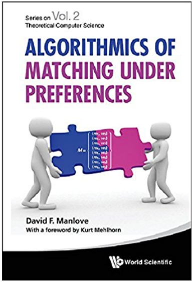

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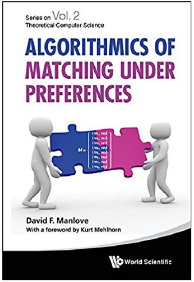

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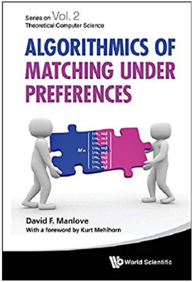

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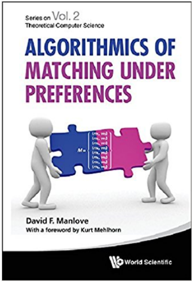

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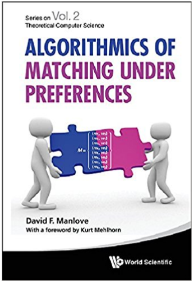

- They are everywhere! (doctors to hospitals, students to universities, kidney exchange, etc)
- *Stability* is the most desired property
- Modularity & Flexibility of CP to solve hard problems?
- Global constraints for stable matching problems?

Matching Under Preferences

Matching Under Preferences

- Assign man to woman
- Every woman has a personnel preference over men
- Every man has a preference list over woman
- Every man/woman is assigned to at most one partner from the opposite sex

Matching Under Preferences

- Assign residents to hospitals
- Every resident has a personnel preference over hospitals
- Each hospital has a preference list over residents
- each hospital has a capacity

Matching Under Preferences

- Assign students to universities
- Every student has a personnel preference over universities
- Each university has a preference list over students
- each university has a capacity

Matching Under Preferences

- Assign workers to firms
- Every worker has a preference list over firms
- Every firm has a preference list over workers
- Every worker w is assigned to a number n_w of firms
- Every firm f is assigned to a number n_f of workers

Matching Under Preferences

Context

- Two sets of agents
- Two sided preferences (complete or incomplete)
- Stable matching M:
  - Capacity constraints satisfied
  - There exists no pair of agents that prefer each other to their situation in M
- Eventually one can have side constraints

Many-to-Many Stable Matching

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- $W = \{w_1, w_2, \ldots, w_{n_W}\}$: a set of workers

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- M is stable if the quota constraints are respected and no pair $\langle w, f \rangle$ has an incentive to deviate from M by being matched together

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- M is stable if the quota constraints are respected and no pair $\langle w, f \rangle$ has an incentive to deviate from M by being matched together
- Solvable in $O(L)$ time
- NP-Hard variants with side constraints

Example

w_0	0 6 5 2 4 1 3
w_1	6 1 4 5 0 2 3
w_2	6 0 3 1 5 4 2
w_3	3 2 0 1 4 6 5
w_4	1 2 0 3 4 5 6
w_5	6 1 0 3 5 4 2
w_6	2 5 0 6 4 3 1

f_0	2 1 6 4 5 3 0
f_1	0 4 3 5 2 6 1
f_2	2 5 0 4 3 1 6
f_3	6 1 2 3 4 0 5
f_4	4 6 0 5 3 1 2
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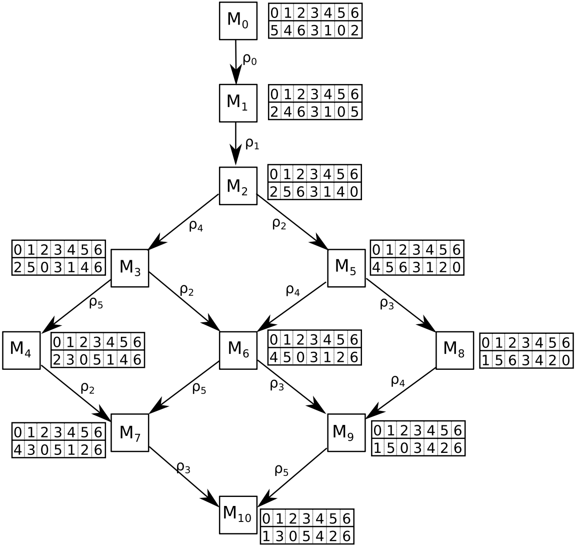
f_6	4 6 2 1 3 0 5

Dominance Relation on Stable Matchings



Rotation

- $M_1 : \langle 0, 2 \rangle, \langle 1, 4 \rangle, \langle 2, 6 \rangle, \langle 3, 3 \rangle, \langle 4, 1 \rangle, \langle 5, 0 \rangle, \langle 6, 5 \rangle$
- $M_2 : \langle 0, 2 \rangle, \langle 1, 5 \rangle, \langle 2, 6 \rangle, \langle 3, 3 \rangle, \langle 4, 1 \rangle, \langle 5, 4 \rangle, \ \langle 6, 0 \rangle$

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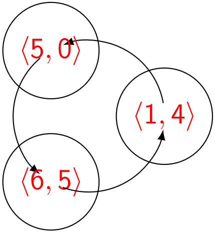

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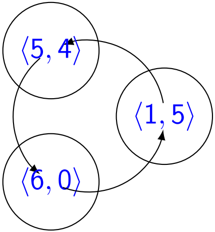

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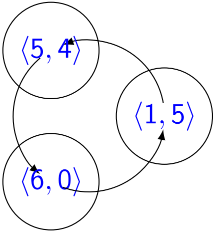

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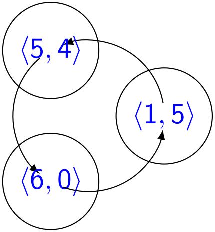

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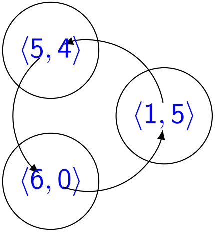

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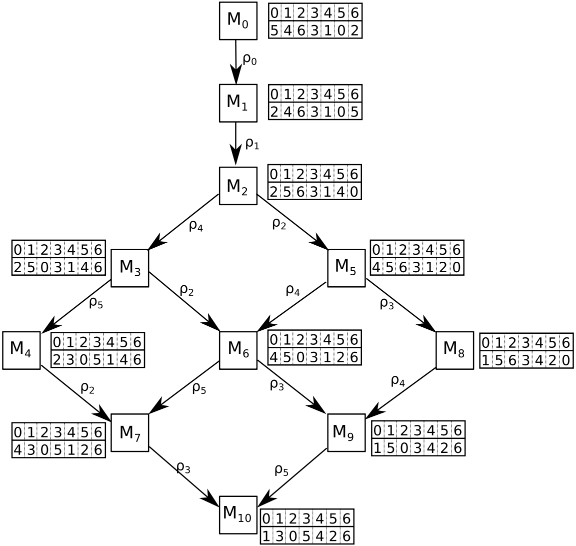
- $\langle 1,4\rangle$ is eliminated by ρ_1
- $\langle 1,5\rangle$ is produced by ρ_1

A Partial Order on Rotations

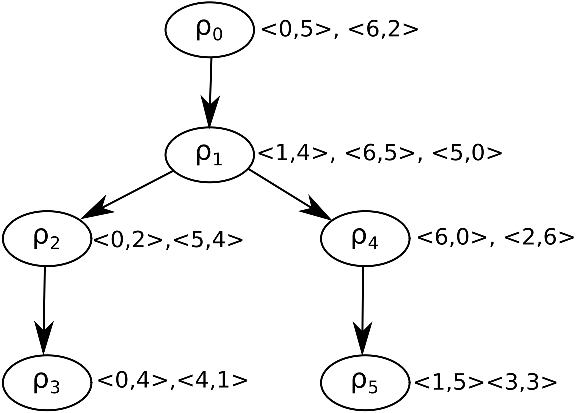
$\rho_1 \prec\prec \rho_2$

- ρ_1 precedes ρ_2 if ρ_1 has to be applied before ρ_2 in every succession of rotation eliminations leading from M_0 to M_z.

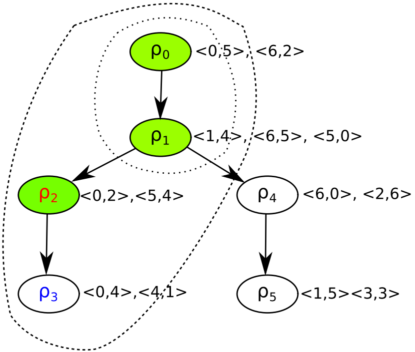
The Partial Order on Rotations



Graph Poset



Closed Subset



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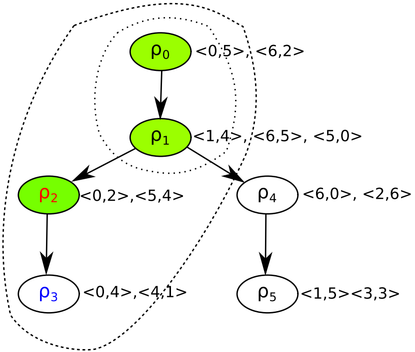

Theorem [Gusfield and Irving, 1989, Bansal et al., 2007]

There is a one-to-one mapping between closed subsets and stable matchings

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- Every non-fixed stable pair $\langle w, f \rangle \notin M_z$ can be eliminated by a unique rotation $\rho_{e_{wf}}$
- Every non-fixed stable pair $\langle w, f \rangle \notin M_0$ can be produced by a unique rotation $\rho_{p_{wf}}$

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- Every non-fixed stable pair $\langle w, f \rangle \notin M_0$ can be produced by a unique rotation $\rho_{p_{wf}}$
- In $O(L)$ time, one can compute:
  - M_0, M_z
  - The fixed, stable and non-stable pairs
  - The set of rotations
  - The graph poset
  - $\rho_{e_{wf}}$ and $\rho_{p_{wf}}$

Lemmas

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2. Else, if $\langle w_i, f_j \rangle \in M_z$, then $\langle w_i, f_j \rangle \in M$ iff $\rho_{p_{ij}} \in S$.
3. Otherwise, $\langle w_i, f_j \rangle \in M$ iff $\rho_{p_{ij}} \in S \wedge \rho_{e_{ij}} \notin S$.

Rotation-based (SAT) Formulation

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- Variables
  - A Boolean variable $x_{i,j}$ for every pair $\langle w_i, f_j \rangle$
  - A Boolean variable r_k for every rotation ρ_k

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    1. if $\langle w_i, f_j \rangle \in FP : x_{i,j}$

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    5. Otherwise, $x_{i,j} == r_{p_{ij}} \wedge \neg r_{e_{ij}}$

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    5. Otherwise, $x_{i,j} == r_{p_{ij}} \wedge \neg r_{e_{ij}}$
- **Easily translated in SAT (Γ)**

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- Let $M2M(I, \mathcal{X}(M2M))$ be the stable matching constraint
- Unit propagation on Γ does not maintain arc consistency
- **Theorem:** Let $\mathcal{D}$ be a domain such that unit propagation is performed without failure on Γ. There exists at least a solution in $\mathcal{D}$ that satisfies Γ.

Arc Consistency

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We know that..

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Arc Consistency

- Idea: use unit propagation as a support check
- Some assignments already have supports
- $O(L^2)$ time

Arc Consistency

Experimental Study

Sex-Equal & Balanced Stable Matching

Experimental Study

Sex-Equal & Balanced Stable Matching

- Let M be a stable marriage
  - C^m_M is the sum of the ranks of each man's partner
  - C^w_M is the sum of the ranks of each woman's partner

Experimental Study

Sex-Equal & Balanced Stable Matching

- Let M be a stable marriage
  - C_M^m is the sum of the ranks of each man's partner
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- Sex-Equal Stable matching: find a stable matching M with the minimum value of $|C_M^m - C_M^w|$

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  - C_M^w is the sum of the ranks of each woman's partner
- Sex-Equal Stable matching: find a stable matching M with the minimum value of $|C_M^m - C_M^w|$
- Balanced stable matching: find a stable matching M with the minimum value of $max\{C_M^m, C_M^w\}$

Experimental Protocol

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- Models:
  - fr: SAT-formula
  - ac: Arc Consistency
  - bc: State-of-the art propagator [Siala and O'Sullivan, 2016]

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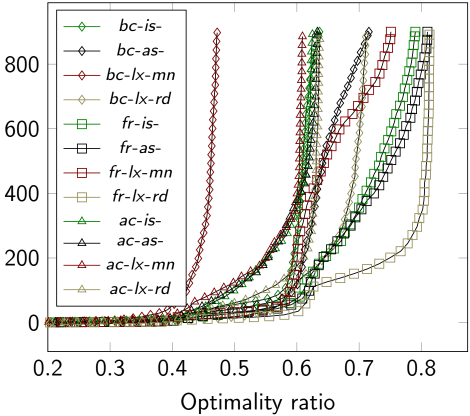
- New challenging benchmarks: `http://siala.github.io/sm/sm.zip`
- 5 randomised runs for every configuration
- 15 minutes cutoff for every run

Sex-Equal Stable Matching: Optimality Evaluation



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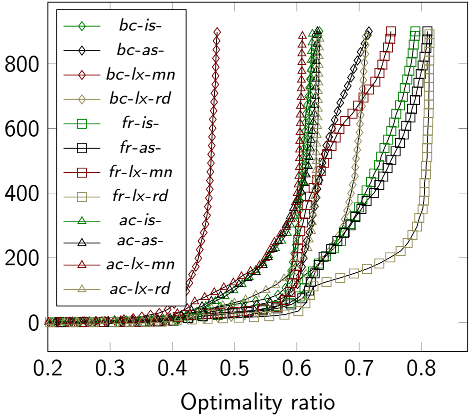
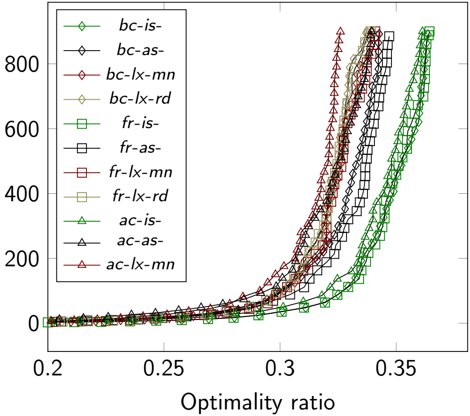
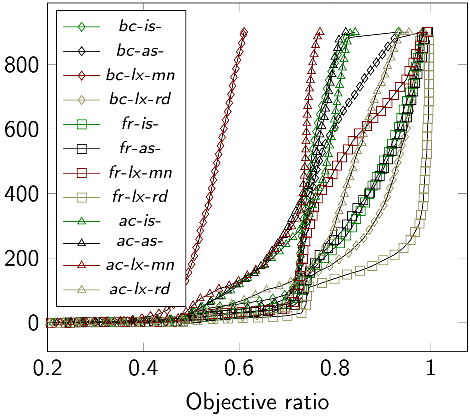


- Clear dominance of the SAT formulation
- Arc Consistency does not pay off

Balanced Stable Matching: Optimality Evaluation

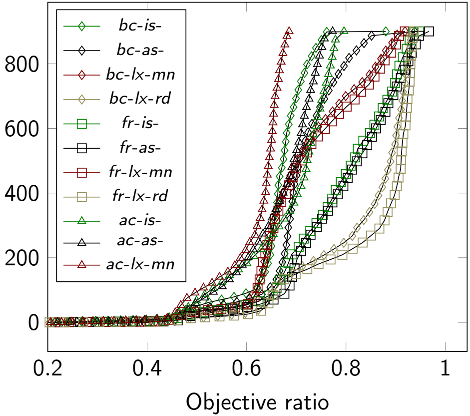


Sex-Equal Stable Matching: Solution Quality



- Better Solutions with the SAT model
- Arc Consistency does not pay off

Balanced Stable Matching: Solution Quality



Conclusions & Future Research

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Take-away message

- No need for implementing a sophisticated global constraint for stability. Use the rotations reformulation!

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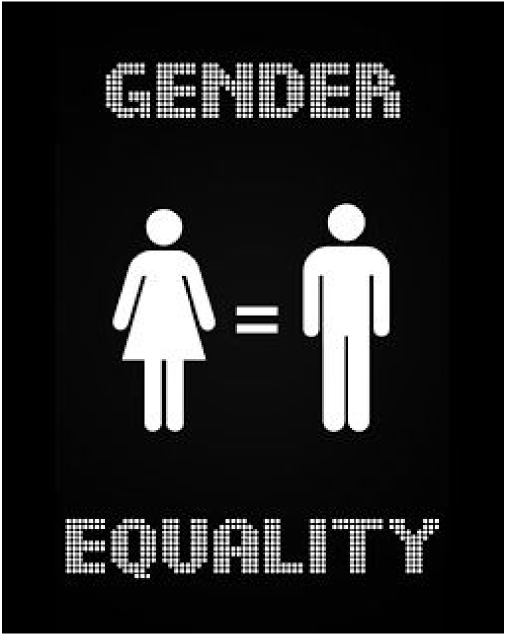
Future Research

- Other applications?
- Stable matching with ties?
- Stable matching with couples?
- One sided preferences?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EQUALITY

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