Centre for
Data Analytics

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From Backdoor Key to Backdoor Completability: Improving a Known Measure of Hardness for the Satisfiable CSP

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Context

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- Understanding instance hardness?

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- Not from a complexity point of view, but from solver point of view

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- The notion of backdoor

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- The notion of backdoor isolates the area where hardness occurs in terms of variables.

Definition (Williams, Gomes, Selman 2003)

Let I be a satisfiable CSP instance, a backdoor is a subset B of variables such that there exists an assignment of the variables in B that makes it easy to find a solution for I.

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Definition (Ruan, Kautz, Horvitz 2004)

Let B be a backdoor for a satisfiable instance I.
Let $v \in B$ and let S a partial solution for $B \setminus \{v\}$.

- v is dependent: exactly one value a in the domain of v such that $S \cup \{a\}$ can be extended to a solution for I.
- key fraction of B: $\frac{\#\text{dependent variables in } B}{|B|}$

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Hypothesis

higher backdoor key fraction $\Leftrightarrow$ harder instance

Limitation of BKF

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- Case when there is only one solution: $\Longrightarrow$ BFK always 1.
- Case when flipping the truth assignment of any variable in the backdoor and still extend the backdoor to a solution: $\Longrightarrow$ BFK always 0.

Characterizing Hardness in Backtracking Search

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- Large search space + small solution space
  $\implies$ A lot of choices made, few of them good
  $\implies$ The solver takes a long time to find a solution.

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Combining both

- Large search space + small solution space
  $\Longrightarrow$ A lot of choices made, few of them good
  $\Longrightarrow$ The solver takes a long time to find a solution.
- Solution space close to the search space
  $\Longrightarrow$ Almost all the choices made are good
  $\Longrightarrow$ solver finds a solution quickly.

Completability Ratio

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Definitions

Let B be a backdoor for I. Completability ratio of B is $\frac{\#completable}{\#partial}$, with:

- $\#partial$ the number of partial solutions for B.
- $\#completable$ the number of partial solutions for B that can be extended to a solution for I.

Backdoor Completability

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average completability ratio of all minimal backdoors of I.

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lower backdoor completability $\Leftrightarrow$ harder instance

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What we want

A class $\mathcal{C}$ of instances, a solver A.

- $\mathcal{C}$ is tractable for $A \rightarrow$ backdoor completability of **all** instances in $\mathcal{C}$ is **high**.
- $\mathcal{C}$ is not tractable for $A \rightarrow$ backdoor completability of **some** instances in $\mathcal{C}$ is **low**.

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Instance classes

- $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \ldots$ where a satisfiable instance is in $\mathcal{C}_p$ if the treewidth of its primal constraint graph is bounded by p.
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Solvers

- $A_1, A_2, \ldots$ where A_q is based on $(q, 1)$-consistency.
- The transition between tractability and hardness is sharp: for any class C_p and solver A_q:
  - $p \leq q \rightarrow A_q$ finds $\mathcal{C}_p$ trivial.
  - $p > q \rightarrow A_q$ fails to solve some instances in $\mathcal{C}_p$.

  so the distinction between low and high completability values should be clear.

Theorem

For any two integers p, q exactly one of the following is true:

1. $p \leq q$ and for every $I \in \mathcal{C}_p$, the backdoor completability of I is 1.
2. $p > q$ and there is an infinite number of instances in $\mathcal{C}_p$ with a backdoor completability exponentially low in the number of variables.

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Proof

1. From the construction of the solvers.
2. For each number n, we build an instance in $\mathcal{C}_p$ with more than n variables and a backdoor completability with regard to A_q asymptotically lower than $\frac{1}{2^n}$.

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  - 1100 instances of order 22
  - Number of “holes” between 192 and 222: peak of difficulty at 204

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- Mistral solver with default configuration
- Standard method to find backdoors

Quasigroup With Holes

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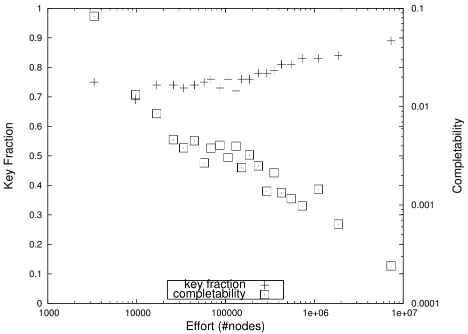

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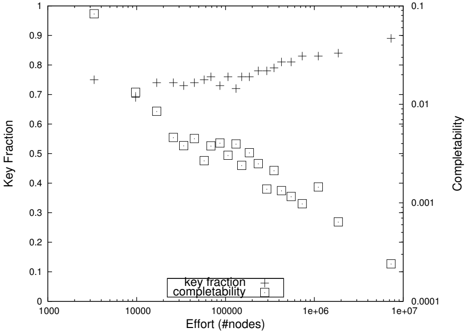
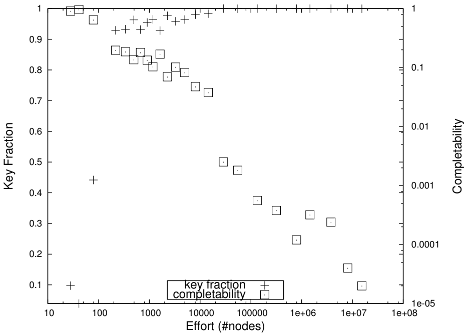


		BD key fraction	BD completability
QWH	Pearson CC	.876	-.943
	RMSE	.053	.037
	MAE	.032	.029

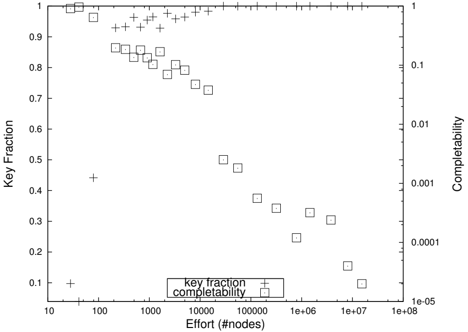
Pearson CC Pearson Correlation Coefficient
RMSE Root Mean Square Error
MAE Mean Absolute Error

Random CSP



Key Fraction
Completability
Effort (#nodes)
key fraction
completability
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
1
0.1
0.01
0.001
0.0001
1e-05
10
100
1000
10000
100000
1e+06
1e+07
1e+08

Random CSP



		BD key fraction	BD completability
Random	Pearson CC	.590	-.975
	RMSE	.186	.051
	MAE	.158	.044

Pearson CC Pearson Correlation Coefficient

RMSE Root Mean Square Error

MAE Mean Absolute Error

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- Generate hard instances?

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Future Research

- Predict hardness? Very expensive to compute : Could it be approximated?
- Generate hard instances?
- Design branching heuristics?

Thank you!