An Introduction to Boolean Satisfiability

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Context : Decision Making

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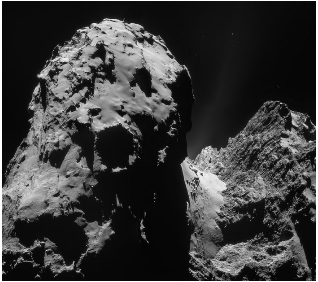

https://homepages.laas.fr/ehebrard/rosetta.html

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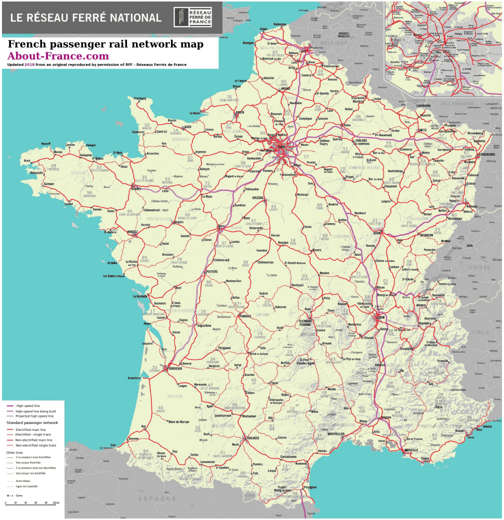
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- Diagnostic decision making: usually as post-processing.

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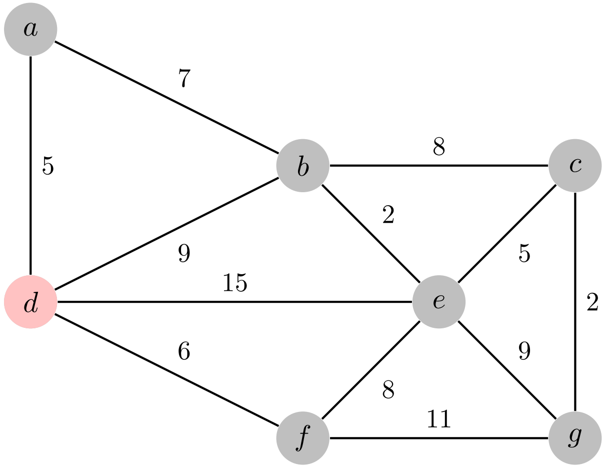
- SAT as an efficient tool for prescriptive decision making
- We focus in this course on the modelling aspect
- Resources for combinatorial optimisation: Many! a good start would be the online course on discrete optimisation `https://www.coursera.org/learn/discrete-optimization`
- Handbook of Satisfiability - Second Edition - IOS Press, 2021

Exemple



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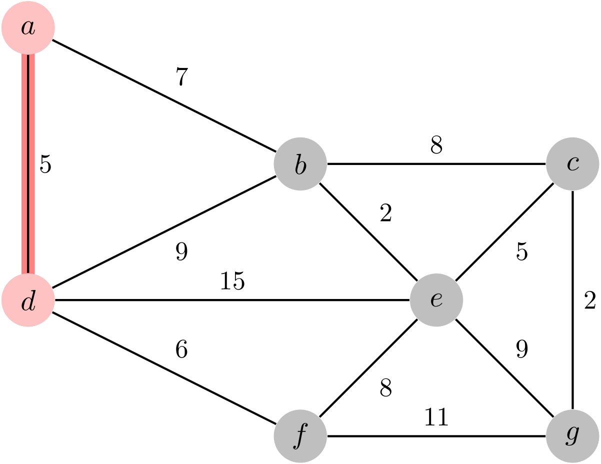

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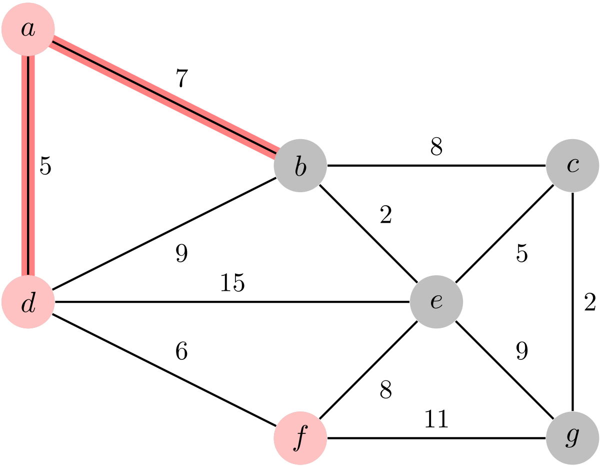

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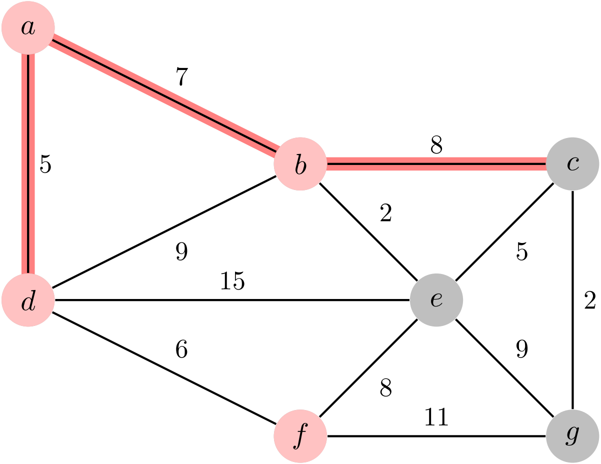

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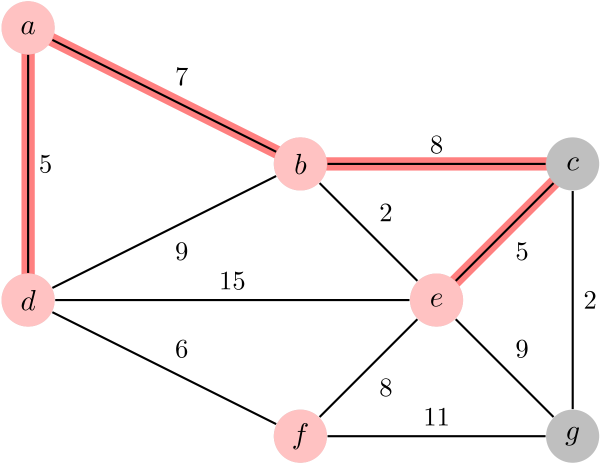

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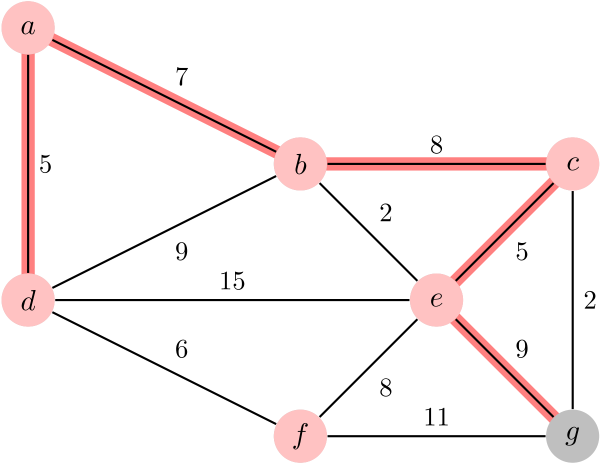

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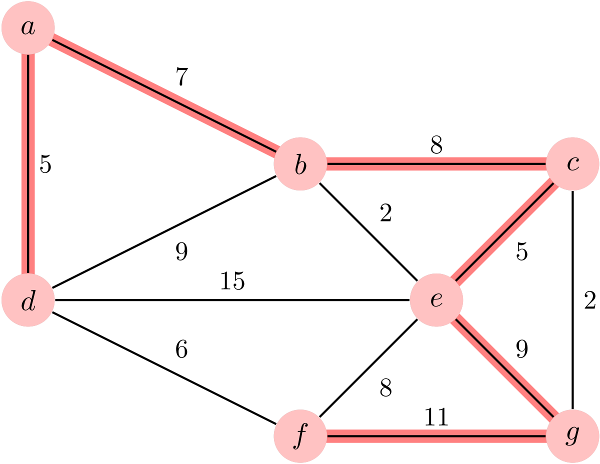

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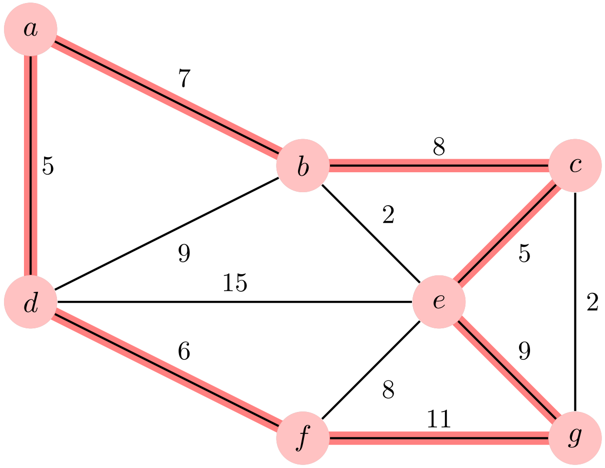

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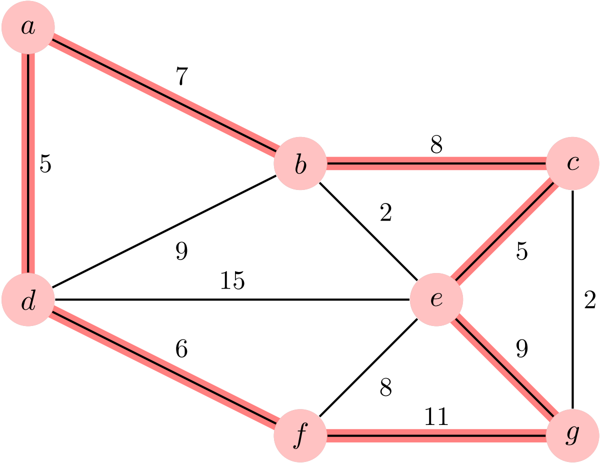

$$-- > Cost : 5 + 7 + 8 + 5 + 9 + 11 + 6 = 53Km$$

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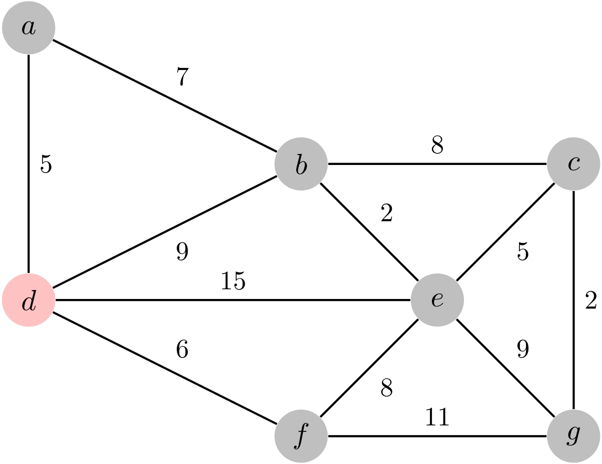

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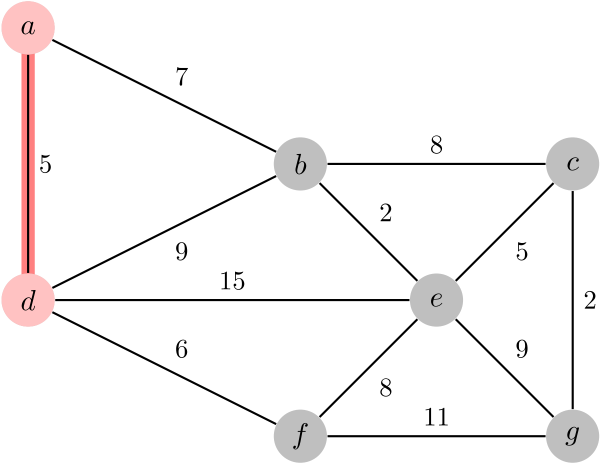

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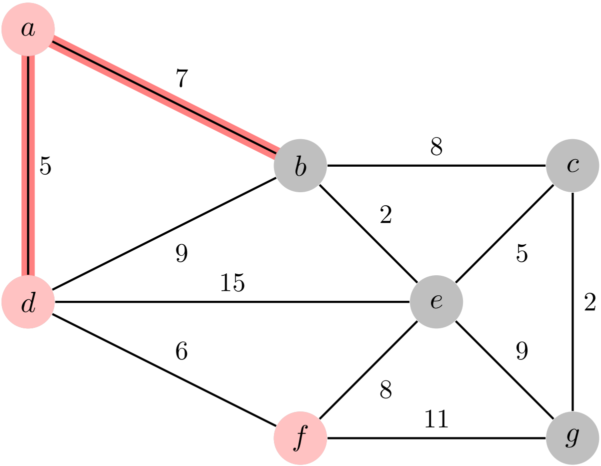

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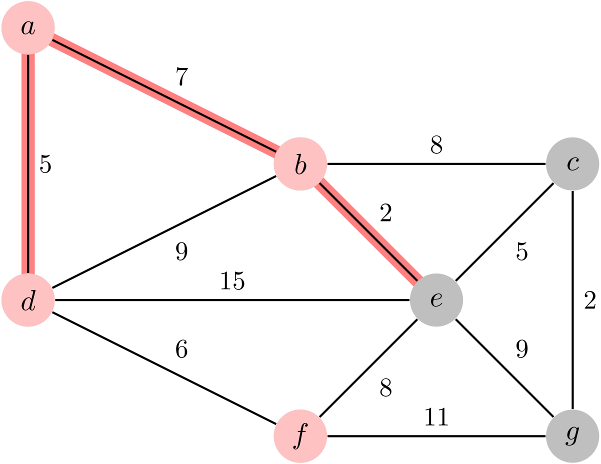

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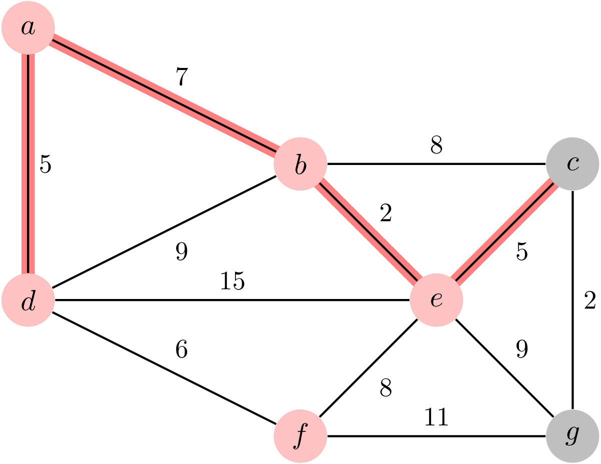

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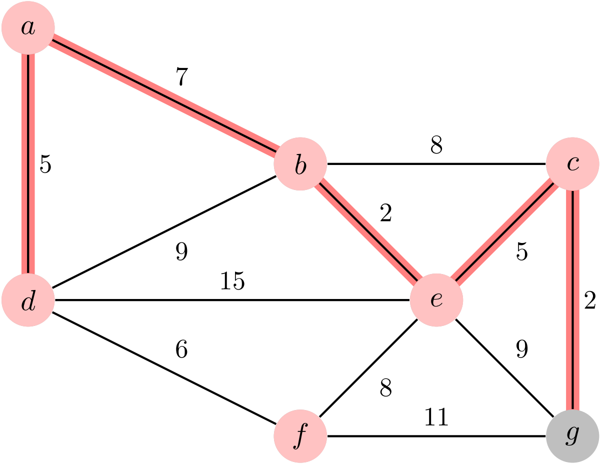

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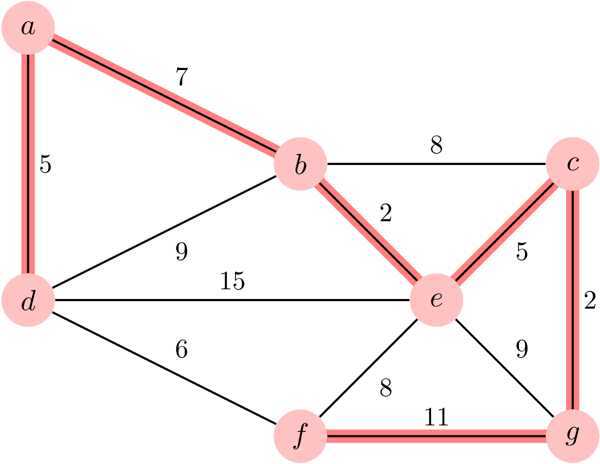

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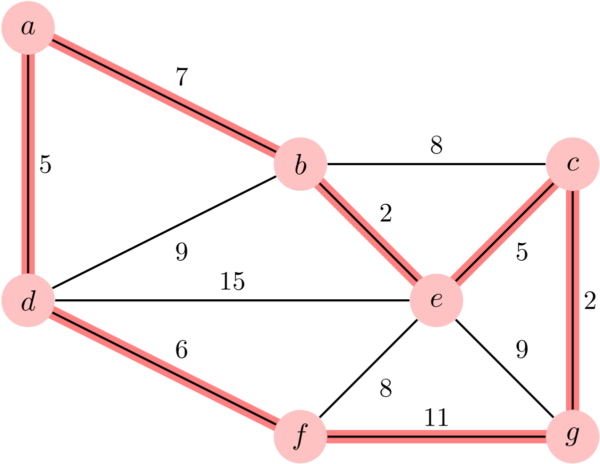

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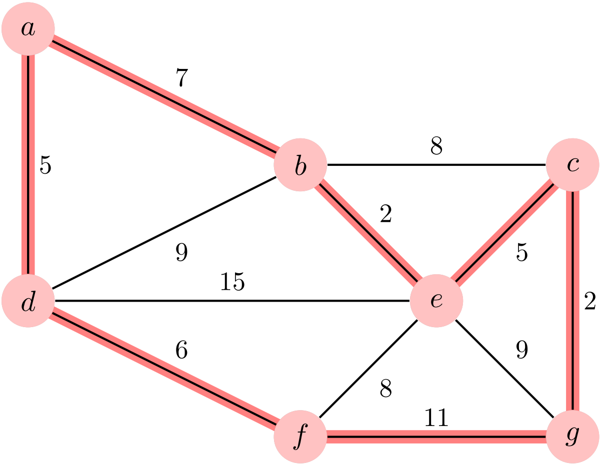

$--> Cost : 5 + 7 + 2 + 5 + 2 + 11 + 6 = 38Km$

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The problem is inherently hard. However, the Concorde algorithm can solve instances up to 86 000 cities!

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1. Adhoc methods
   1. Specific exact algorithm
   2. Heuristic method
   3. Meta-heuristic (genetic algorithms, ant colony, ..)
2. Declarative Approaches
   1. (Mixed) Integer Programming,
   2. Constraint Programming
   3. Boolean Satisfiability (SAT)
   4. ...

Why Declarative Approaches?

- They are problem independent! The user models the problem in a specific language and the solver does the job!
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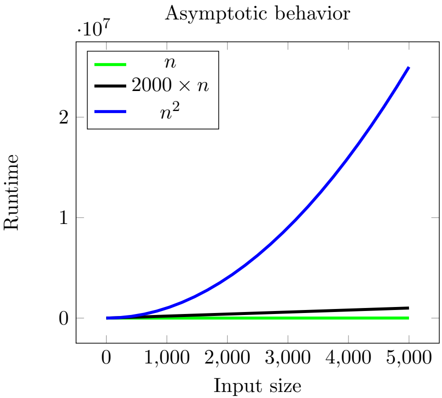

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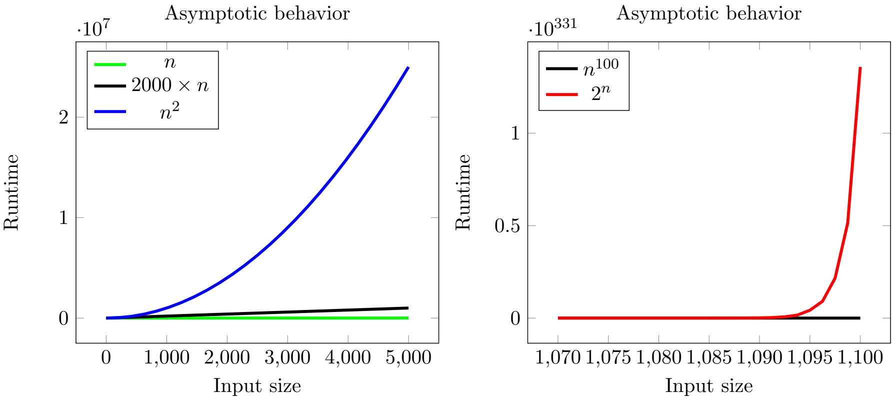

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- What if we don't know if a problem has a polynomial time algorithm?

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- For many Problems in NP, we don't know if a polynomial time algorithm exists. Is P=NP?

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Given a set of Boolean variables $x_1, \ldots x_n$ and a CNF formula Φ over $x_1, \ldots x_n$, the Boolean Satisfiability problem (SAT) is to find an assignment of the variables that satisfies all the clauses.

Example

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$x \lor \neg y \lor z$

$\neg x \lor \neg z$

$y \lor w$

$\neg w \lor \neg x$

Example

$x \vee \neg y \vee z$

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A possible solution:

$$x \leftarrow 1; y \leftarrow 1; z \leftarrow 0; w \leftarrow 0$$

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- Huge practical improvements in the past 2 decades or so

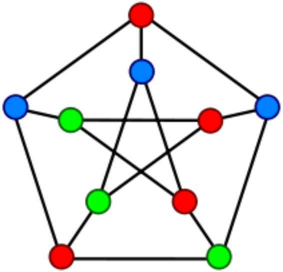
Examples of Applications

- AI Planning
- Scheduling
- Software verification
- Machine learning
  - Robustness
  - Synthesis
  - Verification
- Mathematical Proofs!
  `https://news.cnrs.fr/articles/the-longest-proof-in-the-history-of-mathematics`
- Timetabling
- ...

Modelling in SAT

The example of Graph Colouring

- Graph Coloring is a well known combinatorial problem that has many applications (in particular in scheduling problems)
- Let $G = (V, E)$ be an undirected graph where V is a set of n vertices and E is a set of m edges. Is it possible to colour the graph with k colours such that no two adjacent nodes share the same colour?



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- Forbid two nodes that share an edge to be coloured with the same colour

$$\forall \{i, j\} \in E, \forall a \in [1, k] : \neg x_i^a \vee \neg x_j^a$$

  (This is a translation of $x_i^a \rightarrow \neg x_j^a$)

The Example of Graph Coloring: The Model Size

What is the (space) size of the model?

- $n \times k$ Boolean variables
- Constraints form 1: n clauses with k literals each
- Constraints form 2: $n \times k^2$ binary clauses
- Constraints form 3: $m \times k$ binary clauses

The Example of Graph Coloring: The Minimization Version

- Propose a method that uses SAT for the minimisation version of the problem. That is, given $G = (V, E)$, we seek to find the minimum value of k to satisfy the colouring requirements.

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  - **Binary search:** Run iteratively $SAT(V, E, z)$ as long as $UB > LB$ where $z = \lceil (UB - LB)/2 \rceil$. If the result is satisfiable, then and $UB \leftarrow z$ otherwise $LB \leftarrow z$

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- Lower bound: one can simply consider 2 as long as there is an edge. A more advanced one is to look for a clique in the graph.
- An alternative approach is to look for valid theoretical bounds in the literature.

Exercices: Circular dinner

- n people are invited to dinner.
- M is a (Boolean) compatibility matrix. That is, $M[i][j] = 1$ iff., i enjoys dinnig with j
- The purpose is to organize a circular dinner such that each person enjoys having dinner with the four closest persons on the table (i.e., neighborhood of distance 2)

Modelling Cardinality Constraints

- A cardinality constraint takes as input a sequence of Boolean variables $[x_1, \ldots, x_n]$ and an integer k and enforces

$$\sum_{1}^{n} x_i \leq k$$

- Cardinality constraints are everywhere!
- There exist many ways in the literature to encode such constraints. See for instance `https://www.carstensinz.de/papers/CP-2005.pdf`

Quadratic encoding for $\sum_1^n x_i = 1$

- At least one constraint:

$$x_1 \vee x_2 \ldots \vee x_n$$

- at most one constraint:

$$\forall i, j : \neg x_i \vee \neg x_j$$

This generates one clause of size n and (n^2) binary clauses without introducing additional variables.

Linear encoding for $\sum_1^n x_i = 1$

A sequence of Boolean variables $[y_1, \ldots, y_n]$ is introduced such that $\forall i \in [1, n], y_i$ is true iff $\sum_{l=1}^{l=i} x_l = 1$. The set of clauses for the encoding is the following:

$$x_1 \vee x_2 \ldots \vee x_n$$

$$y_n^1$$

$$\forall i \in [1, n-1] : y_i \rightarrow y_{i+1}$$

$$\forall i \in [1, n-1] : y_i \rightarrow \neg x_{i+1}$$

$$\forall i \in [1, n] : x_i \rightarrow y_i$$

Size: n new variables, 1 $n-$ary clause and $3 \times n$ binary clauses,

Encoding for $\sum_1^n x_i \geq k$

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- Vertical relationship: $\forall i \in [1,n], \forall z \in [1,k-1] : y_i^{z+1} \rightarrow y_i^z$

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- Do not Increment: $\neg y_{i-1}^z \wedge \neg x_i \rightarrow \neg y_i^z$

Encoding for $\sum_1^n x_i \geq k$

Size of the encoding:

- $\Theta(n \times k)$ variables
- $\Theta(n + k)$ unary clauses
- $\Theta(n \times k)$ binary clauses
- $\Theta(n \times k)$ ternary clauses

Encoding for $\sum_1^n x_i = k$?

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- Encode $\sum_1^n x_i \geq k+1$
- Add y_n^k
- Replace y_n^{k+1} by $\neg y_n^{k+1}$
- The size of the encoding is the same as $\sum_1^n x_i \geq k$ (asymptotically)

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- Encode $\sum_1^n x_i \leq b$
- $\sum_1^n x_i \geq a$ with the same additional variables
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- Check the MaxSAT competition

The Example of Graph Coloring: A Possible MaxSAT Model

Let $G = (V, E)$ be an undirected graph where V is the set of vertices and E is the set of edges. In the (decision version of the) graph colouring problem, we are given k colours to colour the graph such that no two adjacent nodes share the same colour.

- Propose a MaxSAT model for the minimisation version of the problem. That is, given $G = (V, E)$, we seek to find the minimum value of k to satisfy the colouring requirements.

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The Example of Graph Coloring: A Possible MaxSAT Model

- We shall extend the previous model:
- Let u_a be a Boolean variable that is True iff. the colour $a \in [1, k]$ is used
- Consider the previous model $SAT(V, E, k)$ with k an upper bound.
- All the previous clauses are hard.
- Add the following hard clauses:

$$\forall i \in [1, n], \forall a \in [1, k] : \neg u_a \rightarrow \neg x_i^a$$

- Eventually we can add implied constraints: $u_a \rightarrow u_{a-1}$
- Then add the soft clauses:

$$\forall a \in [1, k] : \neg u_a$$

- A MaxSAT Optimal solution satisfies all the hard coloring clauses (valid colouring) and maximizes the number of non used colours.

Extensions: Quantified Boolean Formula (QBF)

- A QBF has the form Q.F , where F is a CNF-SAT formulae, and Q is a sequence of quantified variables ($\forall x$ or $\exists x$).
- Example $\forall x, \exists y, \exists z, (x \vee \neg y) \wedge (\neg y \vee z)$
- QBF Solver Competition: `https://www.qbflib.org/solvers_list.php`

Extensions: Satisfiability Modulo Theories (SMT)

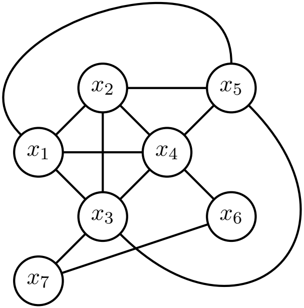
- SMT extends SAT by allowing higher level constraints
- Such constraints belong to certain theories
- Examples of theories include linear integer arithmetic, linear real arithmetic, difference logic, etc
- Check the SAT/SMT summer schools `http://satassociation.org/sat-smt-school.html`

Exercise: SAT for Machine Learning

- Let $F = [f_1, \dots f_k]$ be a set of k features and $E = [e_1, \ \dots e_n]$ a set of n examples.
- We want to build adecision tree
- Task1: Propose a model for the topology of the tree
- Task 2: Extend the model to make sure that each example is well classified
- Task 3: Adapt the model to maximize the accuracy of the model

Exercise: Clique

A clique in a graph $G(V, E)$ (where V is the set of vertices and E is the set of edges). A clique in G is a set of vertices $C \subseteq V$ such that $\forall a, b \in C, \{a, b\} \in E$. For examples, in the example below: $\{x_1, x_2, x_3, x_4, x_5\}$ is a clique and $\{x_3, x_6, x_7\}$ is not a clique.



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- A possible solution:
  - x_i true iff v_i is in the clique
  - For each $\{i, j\} \notin E$:

$$\neg x_i \vee \neg x_j$$

  - Clique size:

$$\sum x_i \geq k$$

  - Implied constraints: If a vertex v_i has less than k edges it shouldn't be part of the clique:

$$\neg x_i$$

MaxSAT

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- Adapt your model into a MaxSAT formulae to find a clique with a maximum size

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  Same model without carnality constraints, without implies constraints, and each x_i is added as a soft clause

Exercise: Shortest Path

Let $G(V, E)$ be a directed graph (where V is the set of vertices and E is the set of directed edges). Suppose that G has a one source $s \in V$ and one sink $o \in V$.

Propose a SAT model to find a path from s to o.

Adapt your model to find a shortest path

Conflict Driven Clause Learning

Modern SAT Solvers: Conflict Driven Clause Learning (CDCL)

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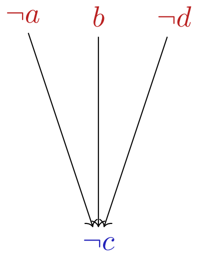
- **Can be seen as a CP Solver (Search, propagation) augmented by clause learning**
- But also :
  - Activity-based branching
  - Lazy data structures (2-Watched Literals)
  - Clause Database Reduction
  - Simplifications
  - Restarts
  - ...

Exercise: Propose a filtering algorithm to prune the variables domain in a given clause

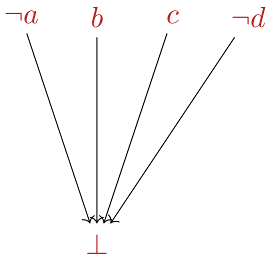
Unit Propagation

Given a clause C of arity n. If $n-1$ literals are false then set the last one to be true.

Example: $(a \vee \neg b \vee \neg c \vee d)$



$\neg a \wedge b \wedge \neg d \Rightarrow \neg c$



$\neg a \wedge b \wedge c \wedge \neg d \Rightarrow \perp$

Algorithm 1: Unit Propagation

Data: A clause C

if *All literals in* C *are false* **then**
    return Failure ;
else
    if *Only one literal* $l \in C$ *is unassigned and the rest are false* **then**
        Make l true ;
    end
end

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- Observe first that propagation happens only in two cases:
  - The clause becomes unit (i.e., all variables except one is instantiated): Propagate the only uninstantiated literal to satisfy the clause
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- Therefore for each clause C, as long as there are two literals non instantiated in C, nothing happens!

Unit Propagation

- Observe first that propagation happens only in two cases:
  - The clause becomes unit (i.e., all variables except one is instantiated): Propagate the only uninstantiated literal to satisfy the clause
  - All literals are instantiated and none of them satisfy the clause
- Therefore for each clause C, as long as there are two literals non instantiated in C, nothing happens!
- The idea of the Two-watched literals is to keep 2 literals for every clause that are not instantiated. Those literals will "watch the clause" and guarantee that no propagation is needed.

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- The idea of the Two-watched literals is to keep 2 literals for every clause that are not instantiated. Those literals will "watch the clause" and guarantee that no propagation is needed.
- If a literal watching a clause C becomes $false$, look for replacement. If no replacement found, then perform propagation

Exercices

- What is the domain of each Boolean variable after propagating the following clauses assuming that a is true and the rest of the variables are unassigned:

$$\neg a \vee g \neg c$$
$$b \vee \neg c \vee g$$
$$a \vee \neg d \vee c$$
$$\neg g \vee a \vee h$$
$$\neg b \vee g \vee d$$
$$b \vee \neg a \vee \neg h$$

- Is the problem satisfiable if $\neg b$ is added? If yes, give a correspondent solution.

Algorithm 2: Two watched Literals (decision d)

▷ Assuming initially that all variables are unassigned and that each clause contains at least 2 literals
▷ For each clause C, $W[C]$ is initialized with a set that contains two variables in C
▷ For each variable x, $B[x]$ is the set of clauses watched by x
▷ d is the latest decision ;

```
S ← {d} ;
while S ≠ ∅ do
    Let x ∈ S ;
    S ← S \ {x} ;
    while B[x] ≠ ∅ do
        Let C ∈ B[x] ;
        if x does not not satisfy C then
            W[C] ← W[C] \ {x} ;
            if ∃x′ ∈ C \ W[C] such that x′ is unassigned then
                W[C] ← W[C] ∪ {x′} ;
                B[x′] ← B[x′] ∪ {C} ;
            else
                Let y ∈ W[C] ;
                if y is not assigned then
                    assign y to a value that satisfies C ;
                    S ← S ∪ {y} ;
                    S ← ∅
                else
                    if y does not satisfy C then
                        return FAILURE ;
                    end
                end
            end
        end
    end
end
```

Learning and Backjumping

Learning and Backjumping

- Definition: Explaining a failure: $l_1 \wedge \ldots \wedge l_n \rightarrow \perp$ where $\neg l_1 \vee \ldots \vee \neg l_n$ is the clause triggering the failure

Learning and Backjumping

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- Definition: Explaining a propagation of l: $l_1 \wedge \ldots \wedge l_n \rightarrow l$ where $\neg l_1 \vee \ldots \vee \neg l_n \vee l$ is the triggering clause

Learning and Backjumping

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- At each conflict learn a new clause as following:

Learning and Backjumping

- Definition: Explaining a failure: $l_1 \wedge \ldots \wedge l_n \rightarrow \bot$ where $\neg l_1 \vee \ldots \vee \neg l_n$ is the clause triggering the failure
- Definition: Explaining a propagation of l: $l_1 \wedge \ldots \wedge l_n \rightarrow l$ where $\neg l_1 \vee \ldots \vee \neg l_n \vee l$ is the triggering clause
- At each conflict learn a new clause as following:
- Start with the explanation from the clause triggering failure in the form of $l_1 \wedge \ldots \wedge l_n \rightarrow \bot$ and let it be the initial explanation

Learning and Backjumping

- Definition: Explaining a failure: $l_1 \wedge \ldots \wedge l_n \rightarrow \perp$ where $\neg l_1 \vee \ldots \vee \neg l_n$ is the clause triggering the failure
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- At each conflict learn a new clause as following:
- Start with the explanation from the clause triggering failure in the form of $l_1 \wedge \ldots \wedge l_n \rightarrow \perp$ and let it be the initial explanation
- While there is more than one literal propagated in the last level in the current explanation, take the lastest one w.r.t. the propagation event, replace it with its explanation from the triggering clause

Learning and Backjumping

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- At each conflict learn a new clause as following:
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- When there is only one literal *uip* propagated in the last level in the current explanation, learn the associated new clause C,

Learning and Backjumping

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- At each conflict learn a new clause as following:
- Start with the explanation from the clause triggering failure in the form of $l_1 \wedge \ldots \wedge l_n \rightarrow \perp$ and let it be the initial explanation
- While there is more than one literal propagated in the last level in the current explanation, take the lastest one w.r.t. the propagation event, replace it with its explanation from the triggering clause
- When there is only one literal *uip* propagated in the last level in the current explanation, learn the associated new clause C, backjump (to the last level of propagated literals in C),

Learning and Backjumping

- Definition: Explaining a failure: $l_1 \wedge \ldots \wedge l_n \rightarrow \perp$ where $\neg l_1 \vee \ldots \vee \neg l_n$ is the clause triggering the failure
- Definition: Explaining a propagation of l: $l_1 \wedge \ldots \wedge l_n \rightarrow l$ where $\neg l_1 \vee \ldots \vee \neg l_n \vee l$ is the triggering clause
- At each conflict learn a new clause as following:
- Start with the explanation from the clause triggering failure in the form of $l_1 \wedge \ldots \wedge l_n \rightarrow \perp$ and let it be the initial explanation
- While there is more than one literal propagated in the last level in the current explanation, take the lastest one w.r.t. the propagation event, replace it with its explanation from the triggering clause
- When there is only one literal uip propagated in the last level in the current explanation, learn the associated new clause C, backjump (to the last level of propagated literals in C), propagate $\neg uip$ via the new clause,

Learning and Backjumping

- Definition: Explaining a failure: $l_1 \wedge \ldots \wedge l_n \rightarrow \perp$ where $\neg l_1 \vee \ldots \vee \neg l_n$ is the clause triggering the failure
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- At each conflict learn a new clause as following:
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- While there is more than one literal propagated in the last level in the current explanation, take the lastest one w.r.t. the propagation event, replace it with its explanation from the triggering clause
- When there is only one literal uip propagated in the last level in the current explanation, learn the associated new clause C, backjump (to the last level of propagated literals in C), propagate $\neg uip$ via the new clause, and continue the exploration

Exercices

- Consider the following formulae

$$
\begin{array}{l}
\neg a \vee g \neg c \\
b \vee \neg c \vee g \\
a \vee \neg d \vee c \\
\neg g \vee a \vee h \\
\neg b \vee g \vee d \\
b \vee \neg a \vee \neg h \\
\neg b \vee a
\end{array}
$$

- Apply the two-watched literals algorithm on the branch d, c, $\neg g$

Conflict Analysis

Algorithm 1: 1-UIP-with-Propagators

$\Psi \leftarrow explain(\bot)$;

while $|\{q \in \Psi \mid level(q) = current\ level\}| > 1$ **do**

- $p \leftarrow \arg\max_q(\{rank(q) \mid level(q) = current\ level \ \wedge \ q \in \Psi\})$;
- $\Psi \leftarrow \Psi \cup \{q \mid q \in explain(p) \wedge level(q) > 0\} \setminus \{p\}$;

return Ψ ;

Conflict Analysis

Algorithm 1: 1-UIP-with-Propagators

```
Ψ ← explain(⊥) ;
while |{q ∈ Ψ | level(q) = current level}| > 1 do
    p ← arg max_q({rank(q) | level(q) = current level ∧ q ∈ Ψ}) ;
    Ψ ← Ψ ∪ {q | q ∈ explain(p) ∧ level(q) > 0} \ {p} ;
return Ψ ;
```

- Why stop with one literal l propagated at the last level ?

Conflict Analysis

Algorithm 1: 1-UIP-with-Propagators

$\Psi \leftarrow explain(\perp)$;
while $|\{q \in \Psi \mid level(q) = current\ level\}| > 1$ **do**
 $p \leftarrow \arg\max_q(\{rank(q) \mid level(q) = current\ level \ \wedge\ q \in \Psi\})$;
 $\Psi \leftarrow \Psi \cup \{q \mid q \in explain(p) \wedge level(q) > 0\} \setminus \{p\}$;
return Ψ ;

- Why stop with one literal l propagated at the last level ?
- **To make sure that when the algorithm backjumps, propagation takes place by making l true**
- When backjumping using a clause that contains more than one literal propagated at the last level, then no propagation can be performed.

Implication Graph

	f		

$\neg a \vee \neg f \vee g$

$\neg a \vee \neg b \vee \neg h$

$a \vee c$

$a \vee \neg i \vee \neg l$

$a \vee \neg k \vee \neg j$

$b \vee d$

$b \vee g \vee \neg n$

$b \vee \neg f \vee n \vee k$

$\neg c \vee k$

$\neg c \vee \neg k \vee \neg i \vee l$

$c \vee h \vee n \vee \neg m$

$c \vee l$

$d \vee \neg k \vee l$

$d \vee \neg g \vee l$

$\neg g \vee n \vee o$

$h \vee \neg o \vee \neg j \vee n$

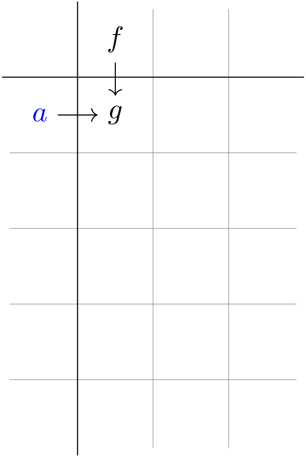
$\neg i \vee j$

$\neg d \vee \neg l \vee \neg m$

$\neg e \vee m \vee \neg n$

$\neg f \vee h \vee i$

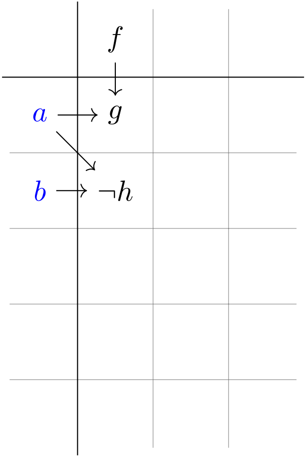
Implication Graph



$\neg a \vee \neg f \vee g$
$\neg a \vee \neg b \vee \neg h$
$a \vee c$
$a \vee \neg i \vee \neg l$
$a \vee \neg k \vee \neg j$
$b \vee d$
$b \vee g \vee \neg n$
$b \vee \neg f \vee n \vee k$
$\neg c \vee k$
$\neg c \vee \neg k \vee \neg i \vee l$

$c \vee h \vee n \vee \neg m$
$c \vee l$
$d \vee \neg k \vee l$
$d \vee \neg g \vee l$
$\neg g \vee n \vee o$
$h \vee \neg o \vee \neg j \vee n$
$\neg i \vee j$
$\neg d \vee \neg l \vee \neg m$
$\neg e \vee m \vee \neg n$
$\neg f \vee h \vee i$

Implication Graph



$\neg a \vee \neg f \vee g$

$\neg a \vee \neg b \vee \neg h$

$a \vee c$

$a \vee \neg i \vee \neg l$

$a \vee \neg k \vee \neg j$

$b \vee d$

$b \vee g \vee \neg n$

$b \vee \neg f \vee n \vee k$

$\neg c \vee k$

$\neg c \vee \neg k \vee \neg i \vee l$

$c \vee h \vee n \vee \neg m$

$c \vee l$

$d \vee \neg k \vee l$

$d \vee \neg g \vee l$

$\neg g \vee n \vee o$

$h \vee \neg o \vee \neg j \vee n$

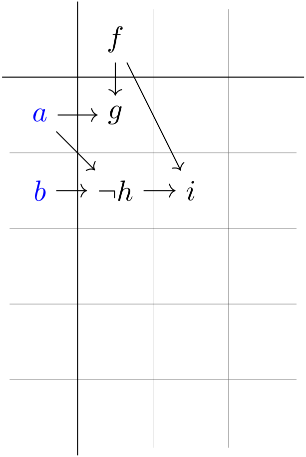
$\neg i \vee j$

$\neg d \vee \neg l \vee \neg m$

$\neg e \vee m \vee \neg n$

$\neg f \vee h \vee i$

Implication Graph



$\neg a \vee \neg f \vee g$

$\neg a \vee \neg b \vee \neg h$

$a \vee c$

$a \vee \neg i \vee \neg l$

$a \vee \neg k \vee \neg j$

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$b \vee g \vee \neg n$

$b \vee \neg f \vee n \vee k$

$\neg c \vee k$

$\neg c \vee \neg k \vee \neg i \vee l$

$c \vee h \vee n \vee \neg m$

$c \vee l$

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$d \vee \neg g \vee l$

$\neg g \vee n \vee o$

$h \vee \neg o \vee \neg j \vee n$

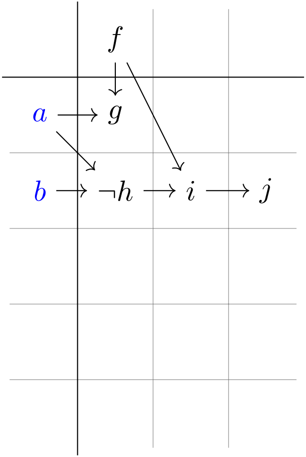
$\neg i \vee j$

$\neg d \vee \neg l \vee \neg m$

$\neg e \vee m \vee \neg n$

$\neg f \vee h \vee i$

Implication Graph



$\neg a \vee \neg f \vee g$

$\neg a \vee \neg b \vee \neg h$

$a \vee c$

$a \vee \neg i \vee \neg l$

$a \vee \neg k \vee \neg j$

$b \vee d$

$b \vee g \vee \neg n$

$b \vee \neg f \vee n \vee k$

$\neg c \vee k$

$\neg c \vee \neg k \vee \neg i \vee l$

$c \vee h \vee n \vee \neg m$

$c \vee l$

$d \vee \neg k \vee l$

$d \vee \neg g \vee l$

$\neg g \vee n \vee o$

$h \vee \neg o \vee \neg j \vee n$

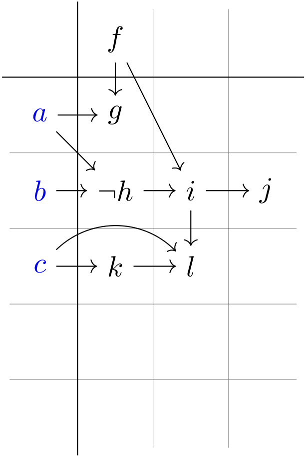
$\neg i \vee j$

$\neg d \vee \neg l \vee \neg m$

$\neg e \vee m \vee \neg n$

$\neg f \vee h \vee i$

Implication Graph



$\neg a \vee \neg f \vee g$

$\neg a \vee \neg b \vee \neg h$

$a \vee c$

$a \vee \neg i \vee \neg l$

$a \vee \neg k \vee \neg j$

$b \vee d$

$b \vee g \vee \neg n$

$b \vee \neg f \vee n \vee k$

$\neg c \vee k$

$\neg c \vee \neg k \vee \neg i \vee l$

$c \vee h \vee n \vee \neg m$

$c \vee l$

$d \vee \neg k \vee l$

$d \vee \neg g \vee l$

$\neg g \vee n \vee o$

$h \vee \neg o \vee \neg j \vee n$

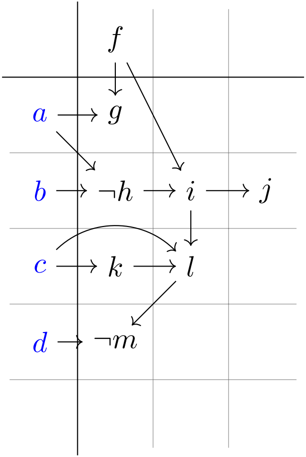
$\neg i \vee j$

$\neg d \vee \neg l \vee \neg m$

$\neg e \vee m \vee \neg n$

$\neg f \vee h \vee i$

Implication Graph



$\neg a \lor \neg f \lor g$

$\neg a \lor \neg b \lor \neg h$

$a \lor c$

$a \lor \neg i \lor \neg l$

$a \lor \neg k \lor \neg j$

$b \lor d$

$b \lor g \lor \neg n$

$b \lor \neg f \lor n \lor k$

$\neg c \lor k$

$\neg c \lor \neg k \lor \neg i \lor l$

$c \lor h \lor n \lor \neg m$

$c \lor l$

$d \lor \neg k \lor l$

$d \lor \neg g \lor l$

$\neg g \lor n \lor o$

$h \lor \neg o \lor \neg j \lor n$

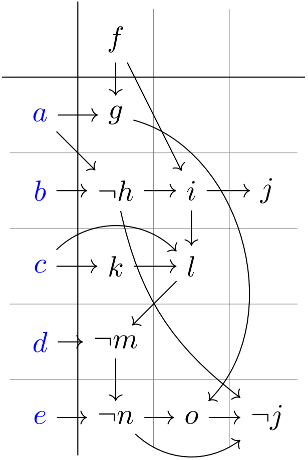
$\neg i \lor j$

$\neg d \lor \neg l \lor \neg m$

$\neg e \lor m \lor \neg n$

$\neg f \lor h \lor i$

Implication Graph



$\neg a \vee \neg f \vee g$

$\neg a \vee \neg b \vee \neg h$

$a \vee c$

$a \vee \neg i \vee \neg l$

$a \vee \neg k \vee \neg j$

$b \vee d$

$b \vee g \vee \neg n$

$b \vee \neg f \vee n \vee k$

$\neg c \vee k$

$\neg c \vee \neg k \vee \neg i \vee l$

$c \vee h \vee n \vee \neg m$

$c \vee l$

$d \vee \neg k \vee l$

$d \vee \neg g \vee l$

$\neg g \vee n \vee o$

$h \vee \neg o \vee \neg j \vee n$

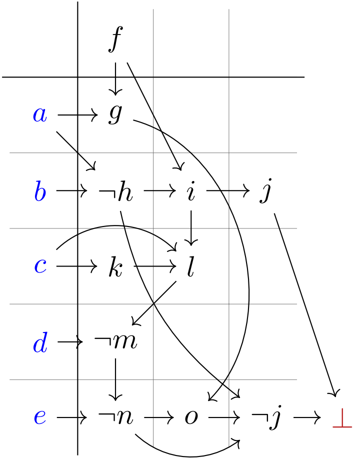
$\neg i \vee j$

$\neg d \vee \neg l \vee \neg m$

$\neg e \vee m \vee \neg n$

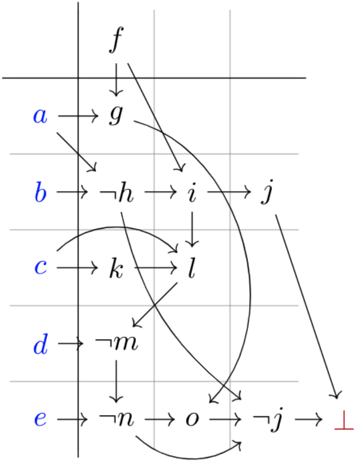
$\neg f \vee h \vee i$

Implication Graph



$\neg a \vee \neg f \vee g$	$c \vee h \vee n \vee \neg m$
$\neg a \vee \neg b \vee \neg h$	$c \vee l$
$a \vee c$	$d \vee \neg k \vee l$
$a \vee \neg i \vee \neg l$	$d \vee \neg g \vee l$
$a \vee \neg k \vee \neg j$	$\neg g \vee n \vee o$
$b \vee d$	$\boxed{h \vee \neg o \vee \neg j \vee n}$
$b \vee g \vee \neg n$	$\neg i \vee j$
$b \vee \neg f \vee n \vee k$	$\neg d \vee \neg l \vee \neg m$
$\neg c \vee k$	$\neg e \vee m \vee \neg n$
$\neg c \vee \neg k \vee \neg i \vee l$	$\neg f \vee h \vee i$

Implication Graph



$\neg a \vee \neg f \vee g$

$\neg a \vee \neg b \vee \neg h$

$a \vee c$

$a \vee \neg i \vee \neg l$

$a \vee \neg k \vee \neg j$

$b \vee d$

$b \vee g \vee \neg n$

$b \vee \neg f \vee n \vee k$

$\neg c \vee k$

$\neg c \vee \neg k \vee \neg i \vee l$

$c \vee h \vee n \vee \neg m$

$c \vee l$

$d \vee \neg k \vee l$

$d \vee \neg g \vee l$

$\neg g \vee n \vee o$

$\boxed{h \vee \neg o \vee \neg j \vee n}$

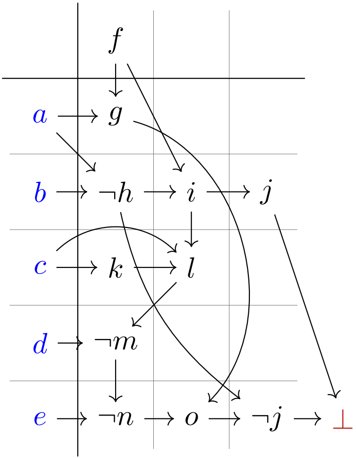
$\neg i \vee j$

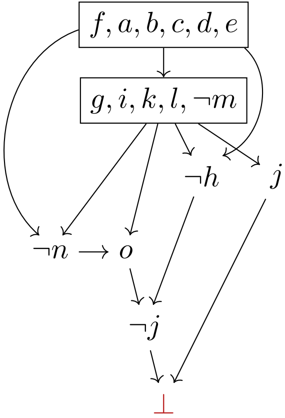
$\neg d \vee \neg l \vee \neg m$

$\neg e \vee m \vee \neg n$

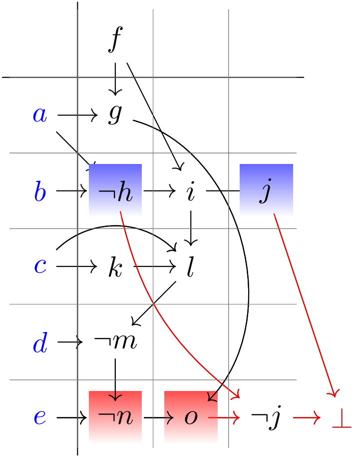
$\neg f \vee h \vee i$

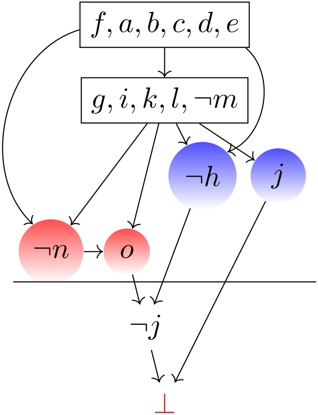
Conflict Analysis



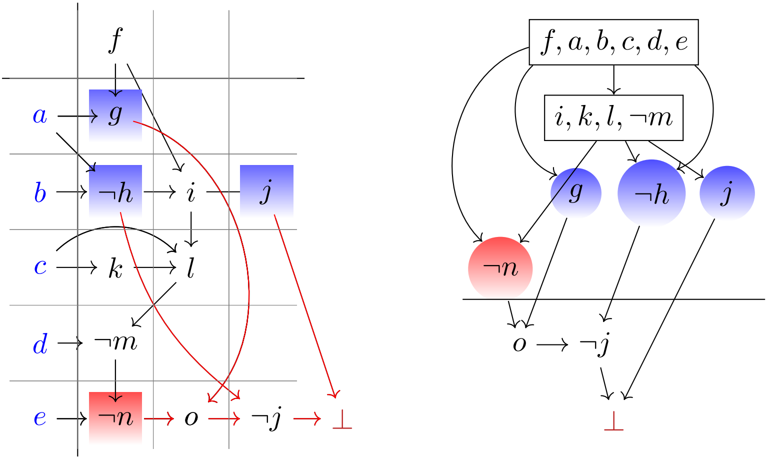


Conflict Analysis

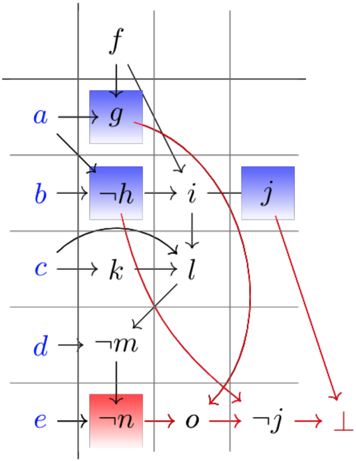


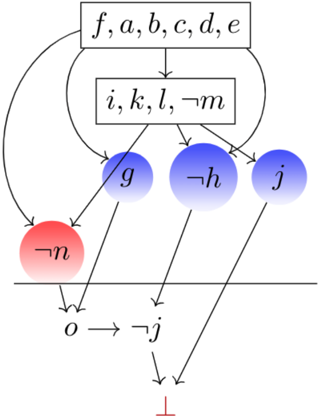


Conflict Analysis

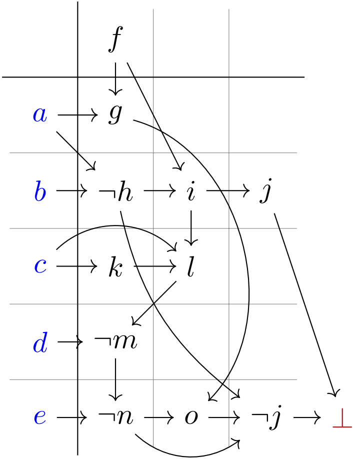


Conflict Analysis





Conflict analysis



$\neg a \vee \neg f \vee g$

$\neg a \vee \neg b \vee \neg h$

$a \vee c$

$a \vee \neg i \vee \neg l$

$a \vee \neg k \vee \neg j$

$b \vee d$

$b \vee g \vee \neg n$

$b \vee \neg f \vee n \vee k$

$\neg c \vee k$

$\neg c \vee \neg k \vee \neg i \vee l$

$c \vee h \vee n \vee \neg m$

$c \vee l$

$d \vee \neg k \vee l$

$d \vee \neg g \vee l$

$\neg g \vee n \vee o$

$h \vee \neg o \vee \neg j \vee n$

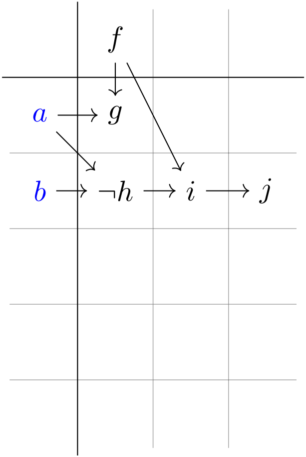
$\neg i \vee j$

$\neg d \vee \neg l \vee \neg m$

$\neg e \vee m \vee \neg n$

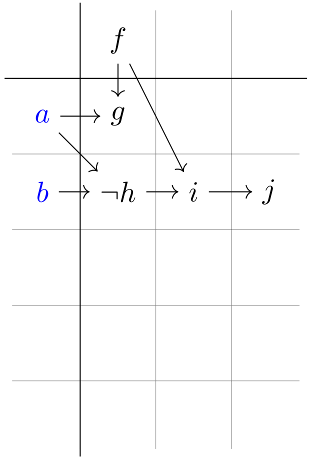
$\neg f \vee h \vee i$

Conflict analysis



$\neg a \vee \neg f \vee g$	$c \vee h \vee n \vee \neg m$
$\neg a \vee \neg b \vee \neg h$	$c \vee l$
$a \vee c$	$d \vee \neg k \vee l$
$a \vee \neg i \vee \neg l$	$d \vee \neg g \vee l$
$a \vee \neg k \vee \neg j$	$\neg g \vee n \vee o$
$b \vee d$	$h \vee \neg o \vee \neg j \vee n$
$b \vee g \vee \neg n$	$\neg i \vee j$
$b \vee \neg f \vee n \vee k$	$\neg d \vee \neg l \vee \neg m$
$\neg c \vee k$	$\neg e \vee m \vee \neg n$
$\neg c \vee \neg k \vee \neg i \vee l$	$\neg f \vee h \vee i$

Conflict analysis



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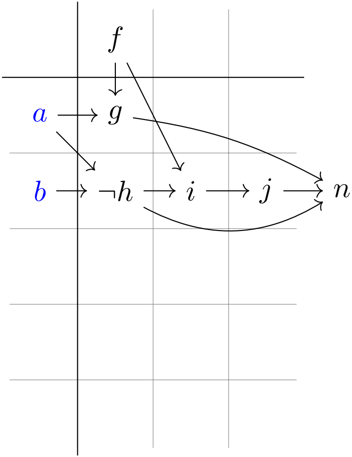

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- Randomization: breaking ties, random decision between k best choices, . . .
- Restarts: Geometric/Luby

Restarts

We find in the literature two common restart policies.

- Geometric restart: $b \times f^{k-1}$ for the k^{th} restart where b is called a base and f is called a factor.
- Luby restarts follow the sequence 1, 1, 2, 1, 1, 2, 4, 1, 1, 2, 1, 1, 2, 4, 8, ... multiplied by a base b. The i^{th} element of the luby sequence ψ_i is defined recursively by the formula:

$$2^{k-1} \; if \; \exists k \in \mathbb{N}, i = 2^k - 1$$

$$\psi_{i-2^{k-1}+1} \; if \; \exists k \in \mathbb{N}, 2^{k-1} \leq i < 2^k - 1$$

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SAT Solvers

- MiniSat: http://minisat.se/
- Glucose: `http://www.labri.fr/perso/lsimon/glucose/`
- LingeLing `http://fmv.jku.at/lingeling`
- Any Solver by Armin Biere
  `http://fmv.jku.at/software/index.html`
- Any winner from past and future SAT competitions:
  https://www.satcompetition.org/

The DIMACS Format (.cnf files)

- A comment line starts with 'c'
- The first non comment line should be in the form p cnf X Y where X is the number of variables and Y is the number of clauses
- For instance, with 4 variables and 3 clauses:
- p cnf 4 3
- Let The list of variables be $x_1, x_2, .., x_n$. The literal x_i is represented by i and the literal $\neg x_i$ is represented by $-i$.
- The clauses are listed line by line where the literals are separated by a space " " and a "0" is placed at the end to indicate the end of the clause

Modelling Exercices

- We want to rebuild the wifi coverage in the GEI department
- A set of geographical locations $G = \{g_1, \ldots g_n\}$ has to be covered
- Potential installations are defined as subsets of G. Each installation covers its elements
- We want to find a full coverage using the minimum number of installations
- Propose a MaxSAT Model

Example

```
p cnf 4 3
2 -4 3 0
1 -2 3 0
-1 -4 -3 0
```

SAT vs CSP

Back to Constraint Programming

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- Mostly solvable by backtracking algorithms (Search and Filtering)

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Value Ordering

'Succeed-first' [Geelen, 1992]:
"Follow the best chances leading to a solution"

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- Filtering (propagation/pruning): inferences based on the current state

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Let C be a constraint and D be a list of domains for the variables in the scope of C.
C is Arc Consistent (AC) iff for every variable x in the scope of C, for every value $v \in D(x)$, there exists an assignment w in D satisfying C in which v is assigned to x

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- If each domain is a singleton, the propagator must be able to check if the assignment corresponds to a solution or not.

CP vs. SAT

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- When should we encode to SAT, when shouldn't we?
- CP vs. SAT: a fundamental difference is the presence of global reasoning in CP and clause learning in SAT

CP vs. SAT : To decompose or not to decompose?

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- **Can we find something that takes advantage from both worlds?** $\rightarrow$ **Clause learning in CP**

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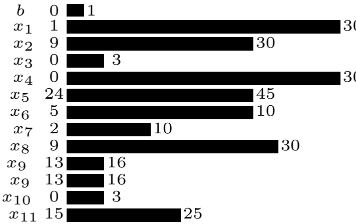

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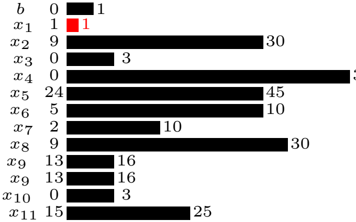

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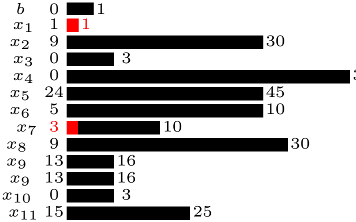

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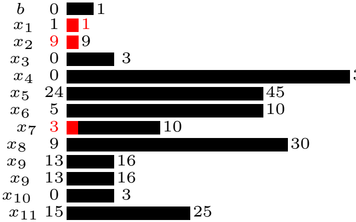

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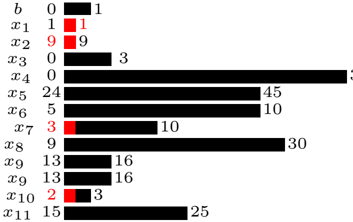

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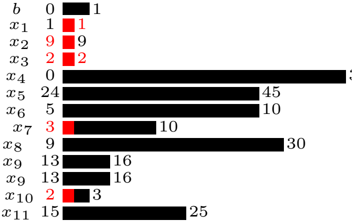

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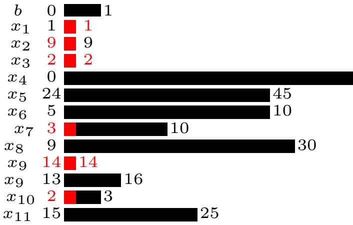

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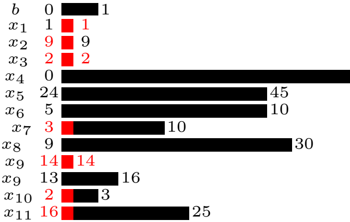

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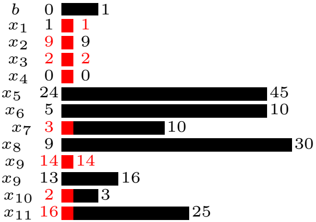

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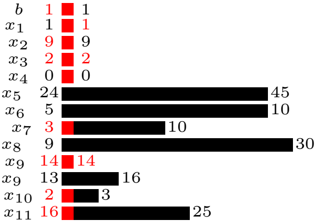

Learning in CP

$[\![x_1 = 1]\!] \rightarrow [\![x_7 \geq 3]\!]$

$[\![x_2 = 9]\!] \rightarrow [\![x_{10} \geq 2]\!] \rightarrow [\![x_{11} \geq 16]\!]$

$[\![x_3 = 2]\!] \rightarrow [\![x_9 = 14]\!] \rightarrow [\![x_{11} \geq 16]\!]$

$[\![x_9 = 14]\!] \rightarrow [\![b = 1]\!]$

$[\![x_4 = 0]\!] \longrightarrow [\![b = 1]\!] \longrightarrow [\![x_6 \geq 7]\!]$

$$
\begin{aligned}
&x_1 + x_7 \geq 4 \wedge \\
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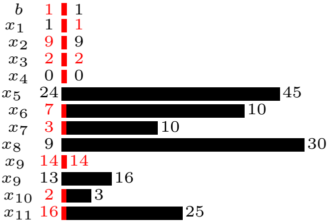

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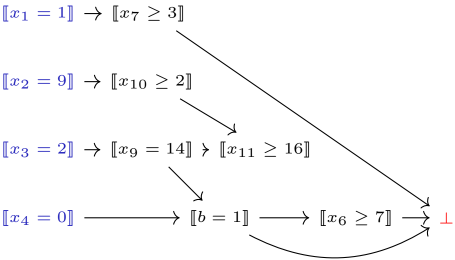

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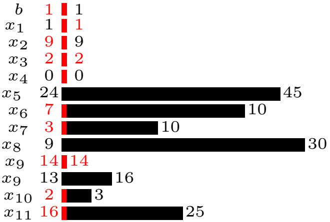

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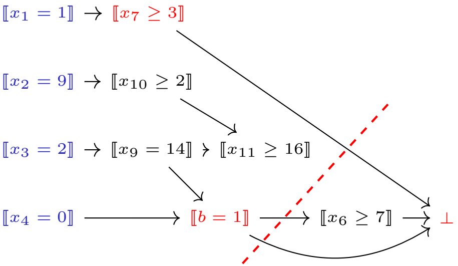

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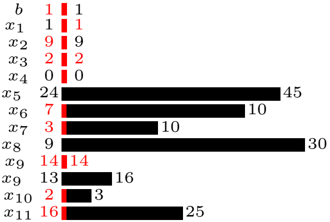

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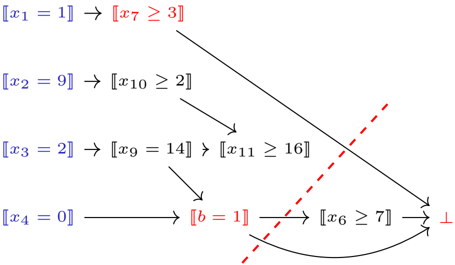

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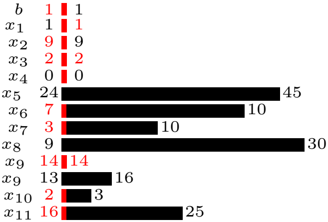

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- Backtrack to level 1

b	0	1
x_1	1	1
x_2	9	30
x_3	0	3
x_4	0	30
x_5	24	45
x_6	5	10
x_7	3	10
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x_9	13	16
x_{10}	0	3
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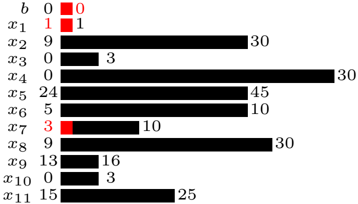
- Backtrack to level 1
- Propagate the learnt clause
- Continue exploration



Conflict analysis

Algorithm 1: 1-UIP-with-Propagators

$\Psi \leftarrow explain(\perp)$;
while $|\{q \in \Psi \mid level(q) = current\ level\}| > 1$ **do**
    $p \leftarrow \arg\max_q(\{rank(q) \mid level(q) = current\ level\ \wedge\ q \in \Psi\})$;
    $\Psi \leftarrow \Psi \cup \{q \mid q \in explain(p) \wedge level(q) > 0\} \setminus \{p\}$;
return Ψ ;

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- To enable clause learning in CP, each propagator must be able to explain its filtering in the form of clauses ("Lazy Clause Generation").
- We distinguish two types of explanations:
  - Explaining Failure
  - Explaining Domain filtering
- Example: Explain the constraint $X \leq Y$ with two scenarios (failure and propagation).

- Let $(x_1, \ldots, x_n)$ be a sequence of Boolean variables, and let d be a positive integer.
- The CARDINALITY$(x_1, \ldots, x_n, d)$ constraint holds iff exactly d variables from the sequence $(x_1, \ldots, x_n)$ are true.
- Write a filtering algorithm for CARDINALITY.
- What is the time complexity?
- Does it enforce arc consistency?
- Explain the CARDINALITY filtering.

Correction

Algorithm 4: CARDINALITY($[x_1, \ldots, x_n], d$)

if $\left|\{x_j \mid \mathcal{D}(x_j) = \{1\}\}\right| > d$ **then**
    $\mathcal{D} \leftarrow \bot$;
if $\left|\{x_j \mid \mathcal{D}(x_j) = \{0\}\}\right| > n - d$ **then**
    $\mathcal{D} \leftarrow \bot$;
if $\left|\{x_j \mid \mathcal{D}(x_j) = \{1\}\}\right| = d$ **then**
    foreach $i \in \{1..n\}$ **do**
        if $\mathcal{D}(x_i) = \{0, 1\}$ **then**
            $\mathcal{D}(x_i) \leftarrow \{0\}$;
else
    if $\left|\{x_j \mid \mathcal{D}(x_j) = \{0\}\}\right| = n - d$ **then**
        foreach $i \in \{1..n\}$ **do**
            if $\mathcal{D}(x_i) = \{0, 1\}$ **then**
                $\mathcal{D}(x_i) \leftarrow \{1\}$;
return $\mathcal{D}$;

Explaining The Cardinality Constraint

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- Failure 1:

$$x^1 \wedge x^2 \wedge \ldots \wedge x^{d+1} \rightarrow \perp$$

Where $D(x^i) = \{1\}$

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$$\neg x^1 \wedge \neg x^2 \wedge \neg x^{n-d+1} \rightarrow \perp$$

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- Explaining the propagating of the value 1: the conjunction of all the assigned variables

Explaining The Cardinality Constraint

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- Failure 2:
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Where $D(x^i) = \{0\}$
- Explaining the propagating of the value 1: the conjunction of all the assigned variables
- Explaining the propagating of the value 0: the conjunction of all the assigned variables

Encoding CSP into SAT

- How to encode the variables' domain ?
- How to encode each constraint into a set of clauses ?

Domain Encoding: Quadratic Encoding

- Suppose that $D(x) = \{v_1, \ldots, v_n\}$

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- That is, $x_i \rightarrow \neg x_j$
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- The number of variables is linear
- The number of clauses is quadratic

Domain Encoding: Linear Encoding

- Suppose that $D(x) = \{1, \ldots, n\}$

Domain Encoding: Linear Encoding

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Domain Encoding: Linear Encoding

- Suppose that $D(x) = \{1, \ldots, n\}$
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- $x_i \rightarrow y_i \wedge \neg y_{i-1}$

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- $y_j \rightarrow y_{j+1}$
- $x_i \rightarrow y_i \wedge \neg y_{i-1}$
- The number of variables is linear in the size of the domain
- The number of clauses is linear. However, some clauses are of arity three

Exercise: Constraint encoding ?

- How to encode the AllDifferent constraint ?
- How to encode $\sum_i X_i \leq k$ (X_i is an integer variable)?
- How to encode $\sum_i a_i \times X_i \leq k$?

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